SIPTA

SIPTA Summer School
12–16 August 2024
Ghent University

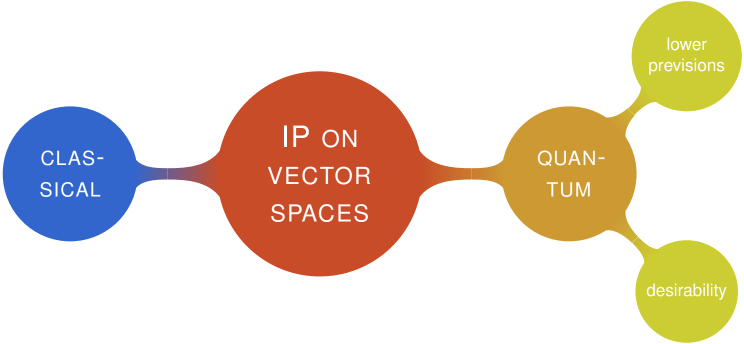
Quantum probability

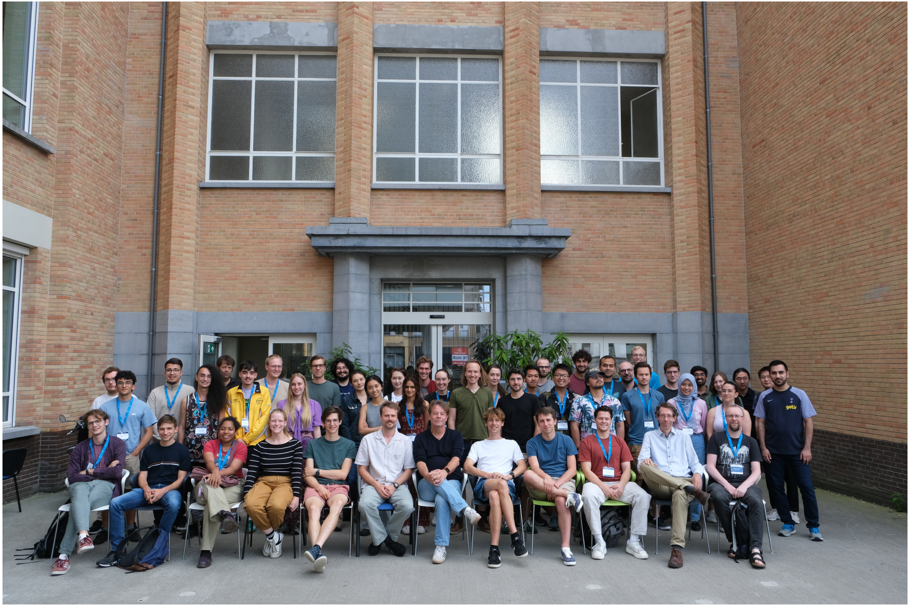
material for the afternoon of day 4

Gert de Cooman

GHENT UNIVERSITY

Foundations Lab
for imprecise probabilities


CLAS-
SICAL
IP ON
VECTOR
SPACES
QUAN-
TUM
lower
previsions
desirability



IMPRECISE PROBABILITY ON NORMED REAL VECTOR SPACES

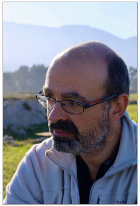
Desirability: pioneers

PETER WILLIAMS

PETER WALLEY

TEDDY SEIDENFELD

SERAFÍN MORAL



ENRIQUE MIRANDA

MARCO ZAFFALON

ERIK QUAEGHEBEUR

GERT DE COOMAN

Desirability: the basics

Options and preferences

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

Desirability: the basics

EXAMPLES

- gambles $f\colon \mathscr{X} \to \mathbb{R}$ on some set $\mathscr{X}$
- indifference classes of gambles on some set $\mathscr{X}$
- Hermitian operators on a complex Hilbert space

Options and preferences

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

A preference order $\rhd$ represents Your preferences between options:

$u \rhd v$ means that You strictly prefer option u over option v.

Desirability: the basics

Options and preferences

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

A preference order $\rhd$ represents Your preferences between options:

$u \rhd v$ means that You strictly prefer option u over option v.

Rationality criteria for preference

Pr1. the relation $\rhd$ is a strict partial ordering: irreflexive and transitive

Pr2. $u \rhd v \Rightarrow u + w \rhd v + w$ for all $u, v, w \in \mathscr{U}$

Pr3. $u \rhd v \Rightarrow \lambda u \rhd \lambda v$ for all $u, v \in \mathscr{U}$ and $\lambda > 0$

Pr4. if $u \succ v$ then also $u \rhd v$ for all $u, v \in \mathscr{U}$

Desirability: the basics

Options and preferences

The option space $\mathscr{U}$ is a real linear space, consisting of options u.

A preference order $\rhd$ represents Your preferences between options:

$u \rhd v$ means that You strictly prefer option u over option v.

Rationality criteria for preference

Pr1. the relation $\rhd$ is a strict partial ordering: irreflexive and transitive

Pr2. $u \rhd v \Rightarrow u + w \rhd v + w$ for all $u, v, w \in \mathscr{U}$

Pr3. $u \rhd v \Rightarrow \lambda u \rhd \lambda v$ for all $u, v \in \mathscr{U}$ and $\lambda > 0$

Pr4. if $u \succ v$ then also $u \rhd v$ for all $u, v \in \mathscr{U}$

Here, $\succ$ is some background preference order, reflecting those minimal preferences You must always have.

The preference order is typically partial, no totality requirement.

Desirability: the basics

The background ordering $\succ$ is completely determined by its cone of positive options

$$\mathscr{U}_{\succ 0} := \{u \in \mathscr{U} : u \succ 0\}.$$

The preference order $\rhd$ is completely determined by the convex cone

$$D := \{u \in \mathscr{U} : u \rhd 0\},$$

as

$$u \rhd v \Leftrightarrow u - v \rhd 0 \Leftrightarrow u - v \in D.$$

Desirability: the basics

The background ordering $\succ$ is completely determined by its cone of positive options

$$\mathscr{U}_{\succ 0} := \{u \in \mathscr{U} : u \succ 0\}.$$

The preference order $\rhd$ is completely determined by the convex cone

$$D := \{u \in \mathscr{U} : u \rhd 0\},$$

as

$$u \rhd v \Leftrightarrow u - v \rhd 0 \Leftrightarrow u - v \in D.$$

Desirable options

A desirable option u is one You (strictly) prefer over the zero option.

We call D Your set of desirable options.

Desirability: the basics

The background ordering $\succ$ is completely determined by its cone of positive options

$$\mathscr{U}_{\succ 0} := \{u \in \mathscr{U} : u \succ 0\}.$$

Coherence criteria for desirability

D1. $0 \notin D$

D2. $u, v \in D \Rightarrow u + v \in D$ for all $u, v \in \mathscr{U}$

D3. $u \in D \Rightarrow \lambda u \in D$ for all $u \in \mathscr{U}$ and $\lambda > 0$

D4. if $u \succ 0$ then also $u \in D$ for all $u \in \mathscr{U}$

Desirability: the basics

The background ordering $\succ$ is completely determined by its cone of positive options

$$\mathscr{U}_{\succ 0} \coloneqq \{u \in \mathscr{U} : u \succ 0\}.$$

Coherence criteria for desirability

D1. $0 \notin D$

D2. $u, v \in D \Rightarrow u + v \in D$ for all $u, v \in \mathscr{U}$

D3. $u \in D \Rightarrow \lambda u \in D$ for all $u \in \mathscr{U}$ and $\lambda > 0$

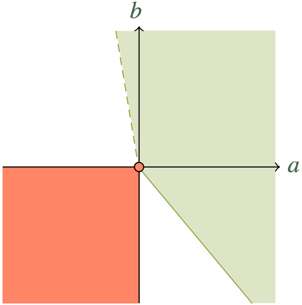
D4. if $u \succ 0$ then also $u \in D$ for all $u \in \mathscr{U}$

A coherent set of desirable options D is a convex cone that includes the positive convex cone $\mathscr{U}_{\succ 0}$ and doesn't contain 0.

We collect all coherent sets of desirable options D in the set $\mathbf{D}$.

Desirability: the basics

$\mathscr{X} = \{a, b\}$



Coherence criteria for desirability

D1. $0 \notin D$

D2. $u, v \in D \Rightarrow u + v \in D$ for all $u, v \in \mathscr{U}$

D3. $u \in D \Rightarrow \lambda u \in D$ for all $u \in \mathscr{U}$ and $\lambda > 0$

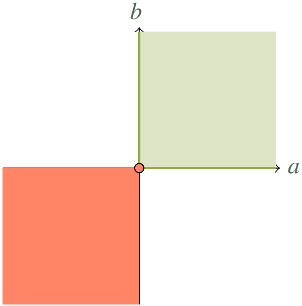
D4. if $u \succ 0$ then also $u \in D$ for all $u \in \mathscr{U}$

A coherent set of desirable options D is a convex cone that includes the positive convex cone $\mathscr{U}_{\succ 0}$ and doesn't contain 0.

We collect all coherent sets of desirable options D in the set $\mathbf{D}$.

Desirability: conservative inference

$\mathscr{X} = \{a, b\}$



Important observation

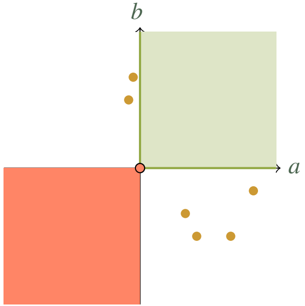
The collection $\mathbf{D}$ of all coherent sets of desirable options is closed under arbitrary non-empty intersections:

$$(\forall i \in I) D_i \in \mathbf{D} \Rightarrow \bigcap_{i \in I} D_i \in \mathbf{D}.$$

The intersection of any non-empty collection of coherent sets of desirable options is still coherent.

Desirability: conservative inference

$$\mathscr{X} = \{a, b\}$$



$$\operatorname{posi}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n > 0, u_k \in V, \lambda_k > 0 \right\}$$

Important observation

The collection $\mathbf{D}$ of all coherent sets of desirable options is closed under arbitrary non-empty intersections:

$$(\forall i \in I) D_i \in \mathbf{D} \Rightarrow \bigcap_{i \in I} D_i \in \mathbf{D}.$$

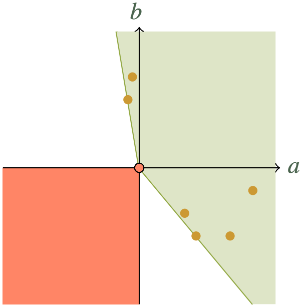
The intersection of any non-empty collection of coherent sets of desirable options is still coherent.

Consistency

An assessment $A \subseteq \mathscr{U}$ is consistent if it is included in some coherent set of desirable options.

Desirability: conservative inference

$\mathscr{X} = \{a, b\}$



$$\operatorname{posi}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n > 0, u_k \in V, \lambda_k > 0 \right\}$$

Important observation

The collection $\mathbf{D}$ of all coherent sets of desirable options is closed under arbitrary non-empty intersections:

$$(\forall i \in I) D_i \in \mathbf{D} \Rightarrow \bigcap_{i \in I} D_i \in \mathbf{D}.$$

The intersection of any non-empty collection of coherent sets of desirable options is still coherent.

Closure (aka Natural extension)

If A is consistent, then

$$\operatorname{cl}_{\mathbf{D}}(A) := \bigcap \{D \in \mathbf{D} : A \subseteq D\} = \operatorname{posi}(A \cup \mathscr{U}_{\succ 0})$$

is the smallest coherent set of desirable options that includes A.

"It's probability theory, Jim, but not as we know it."

Archimedean models: pioneers


BRUNO DE FINETTI


PETER WILLIAMS


PETER WALLEY


MATTHIAS TROFFAES


MARCO ZAFFALON


ENRIQUE MIRANDA


GERT DE COOMAN

International Journal of Approximate Reasoning 140 (2022) 255–281

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Coherent and Archimedean choice in general Banach spaces

Gert de Cooman

Ghent University, Foundations Lab for Imprecise Probabilities, Technologiepark-Zwijnaarde 125, 9052 Zwijnaarde, Belgium

ARTICLE INFO

Article history:
Received 13 April 2021
Received in revised form 9 July 2021
Accepted 13 September 2021
Available online 19 October 2021

Keywords:
Choice function
Set of desirable option sets
Coherence
Archimedeanity
Representation

ABSTRACT

I introduce and study a new notion of Archimedeanity for binary and non-binary choice between options that live in an abstract Banach space, through a very general class of choice models, called sets of desirable option sets. In order to be able to bring an important diversity of contexts into the fold, amongst which choice between horse lottery options, I pay special attention to the case where these linear spaces don't include all 'constant' options. I consider the frameworks of conservative inference associated with Archimedean (and coherent) choice models, and also pay quite a lot of attention to representation of general (non-binary) choice models in terms of the simpler, binary ones. The representation theorems proved here provide an axiomatic characterisation for, amongst many other choice methods, Levi's E-admissibility and Walley–Sen maximality.

Archimedean models: the basics

Structural assumptions

The option space $\mathscr{U}$, provided with a norm $\|\bullet\|_{\mathscr{U}}$, is a normed linear space.

The norm $\|\bullet\|_{\mathscr{U}}$ induces a metric topology on $\mathscr{U}$, with interior operator Int and closure operator Cl.

A real functional $\Gamma\colon \mathscr{U} \to \mathbb{R}$ is bounded if its operator norm $\|\Gamma\|_{\mathscr{U}^\circ}$ is:

$$\|\Gamma\|_{\mathscr{U}^\circ} := \sup_{u \in \mathscr{U} \setminus \{0\}} \frac{|\Gamma(u)|}{\|u\|_{\mathscr{U}}} < +\infty.$$

Properties of a norm $\|\bullet\|_{\mathscr{U}}$:

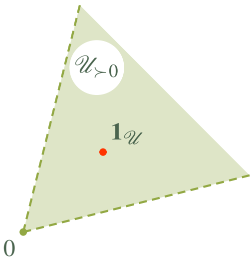
(i) $\|u\|_{\mathscr{U}} \geq 0$;

(ii) $\|u\|_{\mathscr{U}} = 0 \Leftrightarrow u = 0$;

(iii) $\|u+v\|_{\mathscr{U}} \leq \|u\|_{\mathscr{U}} + \|v\|_{\mathscr{U}}$;

(iv) $\|\lambda u\|_{\mathscr{U}} = |\lambda| \|u\|_{\mathscr{U}}$.

Archimedean models: the basics



Structural assumptions

The option space $\mathscr{U}$, provided with a norm $\|\bullet\|_{\mathscr{U}}$, is a normed linear space.

The norm $\|\bullet\|_{\mathscr{U}}$ induces a metric topology on $\mathscr{U}$, with interior operator Int and closure operator Cl.

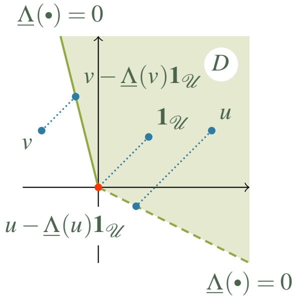
A real functional $\Gamma\colon \mathscr{U} \to \mathbb{R}$ is bounded if its operator norm $\|\Gamma\|_{\mathscr{U}^\circ}$ is:

$$\|\Gamma\|_{\mathscr{U}^\circ} := \sup_{u \in \mathscr{U} \setminus \{0\}} \frac{|\Gamma(u)|}{\|u\|_{\mathscr{U}}} < +\infty.$$

Take as unit element $\mathbf{1}_{\mathscr{U}}$ any element in the interior of $\mathscr{U}_{\succ 0}$:

$$\mathbf{1}_{\mathscr{U}} \in \mathrm{Int}(\mathscr{U}_{\succ 0})$$

Archimedean models: buying and selling price functionals



Other ways to characterise Your preferences?

Buying price functional:

$$\underline{\Lambda}_D(u) := \sup\{\alpha \in \mathbb{R} \colon u - \alpha\mathbf{1}_{\mathscr{U}} \in D\} \text{ for all } u \in \mathscr{U}$$

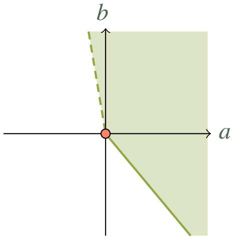
Selling price functional:

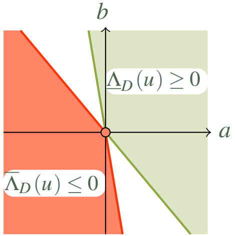
$$\overline{\Lambda}_D(u) := \inf\{\beta \in \mathbb{R} \colon \beta\mathbf{1}_{\mathscr{U}} - u \in D\} \text{ for all } u \in \mathscr{U}$$

Conjugacy:

$$\overline{\Lambda}_D(u) = -\underline{\Lambda}_D(-u) \text{ for all } u \in \mathscr{U}$$

Archimedean models: buying and selling price functionals





Other ways to characterise Your preferences?

Buying price functional:

$$\underline{\Lambda}_D(u) := \sup\{\alpha \in \mathbb{R} \colon u - \alpha \mathbf{1}_{\mathscr{U}} \in D\} \text{ for all } u \in \mathscr{U}$$

Selling price functional:

$$\overline{\Lambda}_D(u) := \inf\{\beta \in \mathbb{R} \colon \beta \mathbf{1}_{\mathscr{U}} - u \in D\} \text{ for all } u \in \mathscr{U}$$

Relation to Your preference model D

$$u \in \operatorname{Int}(D) \Leftrightarrow \underline{\Lambda}_D(u) > 0 \text{ and } u \in \operatorname{Cl}(D) \Leftrightarrow \underline{\Lambda}_D(u) \geq 0$$

The real functional $\underline{\Lambda}_D$ characterises D up to its topological boundary.

Archimedean models: buying and selling price functionals

Other ways to characterise Your preferences?

Buying price functional:

$$\underline{\Lambda}_D(u) := \sup\{\alpha \in \mathbb{R} \colon u - \alpha \mathbf{1}_{\mathscr{U}} \in D\} \text{ for all } u \in \mathscr{U}$$

Selling price functional:

$$\overline{\Lambda}_D(u) := \inf\{\beta \in \mathbb{R} \colon \beta \mathbf{1}_{\mathscr{U}} - u \in D\} \text{ for all } u \in \mathscr{U}$$

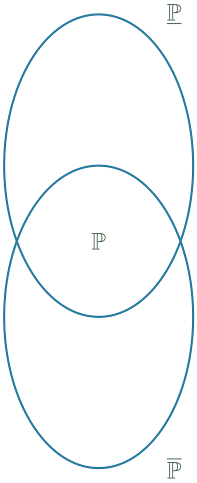
Relation to Your preference model D

$$u \in \operatorname{Int}(D) \Leftrightarrow \underline{\Lambda}_D(u) > 0 \text{ and } u \in \operatorname{Cl}(D) \Leftrightarrow \underline{\Lambda}_D(u) \geq 0$$

The real functional $\underline{\Lambda}_D$ characterises D up to its topological boundary.

$$u \succ v \Leftrightarrow \underline{\Lambda}(u - v) > 0.$$

Archimedean models: coherent (lower and upper) previsions



Coherent lower prevision

A real functional $\underline{P} \colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if there is some coherent set of desirable options D such that $\underline{P} = \underline{\Lambda}_D$.

Coherent upper prevision

A real functional $\overline{P} \colon \mathscr{U} \to \mathbb{R}$ is a coherent upper prevision if and only if there is some coherent set of desirable options D such that $\overline{P} = \overline{\Lambda}_D$.

Coherent prevision

A real functional $P \colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if there is some coherent set of desirable options D such that $P = \underline{\Lambda}_D = \overline{\Lambda}_D$.

Archimedean models: coherent (lower and upper) previsions

Characterisation

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if

L1. $\underline{P}(u+v) \geq \underline{P}(u)+\underline{P}(v)$ for all $u,v \in \mathscr{U}$

L2. $\underline{P}(\lambda u) = \lambda \underline{P}(u)$ for all $u \in \mathscr{U}$ and all real $\lambda > 0$

L3. $\|\underline{P}\|_{\mathscr{U}^\circ} < +\infty$

L4. $\underline{P}(u+\alpha\mathbf{1}_{\mathscr{U}}) = \underline{P}(u)+\alpha$ for all $u \in \mathscr{U}$ and all real α

L5. if $u \succ v$ then $\underline{P}(u) \geq \underline{P}(v)$ for all $u,v \in \mathscr{U}$

A real functional $P\colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if

P1. $P(u+v) = P(u)+P(v)$ for all $u,v \in \mathscr{U}$

P2. $\|P\|_{\mathscr{U}^\circ} < +\infty$

P3. $P(\mathbf{1}_{\mathscr{U}}) = 1$

P4. if $u \succ 0$ then $P(u) \geq 0$ for all $u \in \mathscr{U}$

Archimedean models: coherent (lower and upper) previsions

Vacuous lower and upper previsions

Price functionals associated with the background cone:

$$\operatorname{Inf}_{\succ} u \coloneqq \underline{\Lambda}_{\mathscr{U}_{\succ 0}}(u) = \sup\{\alpha \in \mathbb{R} \colon u \succ \alpha \mathbf{1}_{\mathscr{U}}\}$$

$$\operatorname{Sup}_{\succ} u \coloneqq \overline{\Lambda}_{\mathscr{U}_{\succ 0}}(u) = \inf\{\alpha \in \mathbb{R} \colon u \prec \alpha \mathbf{1}_{\mathscr{U}}\}$$

Archimedean models: coherent (lower and upper) previsions

Vacuous lower and upper previsions

Price functionals associated with the background cone:

$$\operatorname{Inf}_{\succ} u := \underline{\Lambda}_{\mathscr{U}_{\succ 0}}(u) = \sup\{\alpha \in \mathbb{R} \colon u \succ \alpha \mathbf{1}_{\mathscr{U}}\}$$

$$\operatorname{Sup}_{\succ} u := \overline{\Lambda}_{\mathscr{U}_{\succ 0}}(u) = \inf\{\alpha \in \mathbb{R} \colon u \prec \alpha \mathbf{1}_{\mathscr{U}}\}$$

Simpler characterisation

A real functional $\underline{P} \colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if

L0. $\underline{P}(u) \geq \operatorname{Inf}_{\succ} u$

L1. $\underline{P}(u+v) \geq \underline{P}(u) + \underline{P}(v)$ for all $u, v \in \mathscr{U}$

L2. $\underline{P}(\lambda u) = \lambda \underline{P}(u)$ for all $u \in \mathscr{U}$ and all real $\lambda > 0$

A real functional $P \colon \mathscr{U} \to \mathbb{R}$ is a coherent prevision if and only if

P0. $P(u) \geq \operatorname{Inf}_{\succ} u$

P1. $P(u+v) = P(u) + P(v)$ for all $u, v \in \mathscr{U}$

Archimedean models: coherent (lower and upper) previsions

Lower envelope theorem

A real functional $\underline{P}\colon \mathscr{U} \to \mathbb{R}$ is a coherent lower prevision if and only if it is the lower envelope of some set $\mathscr{M}$ of coherent previsions:

$$\underline{P}(u) = \inf\{P(u)\colon P \in \mathscr{M}\} \text{ for all } u \in \mathscr{U}.$$

In that case, the largest such set is the convex and (weak*)-closed

$$\mathscr{M}(\underline{P}) := \{P\colon (\forall u \in \mathscr{U})P(u) \geq \underline{P}(u)\}.$$

This is an instance of the Hahn–Banach Theorem.

The vacuous lower prevision $\mathrm{Inf}_{\succ}$ is the lower envelope of the set of all coherent previsions $\mathbb{P}$.

Archimedean models: coherent (lower and upper) previsions

Related orderings

The weak dominance ordering

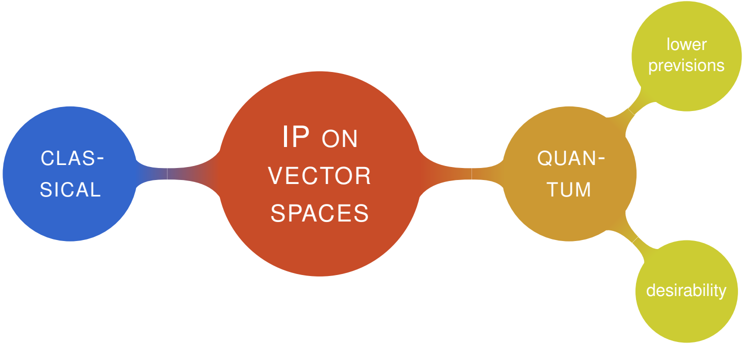
$$u > 0 \Leftrightarrow \mathrm{Inf}_{\succ} u \geq 0 \text{ and } u \neq 0$$

and the strong dominance ordering

$$u \gtrdot 0 \Leftrightarrow \mathrm{Inf}_{\succ} u > 0.$$

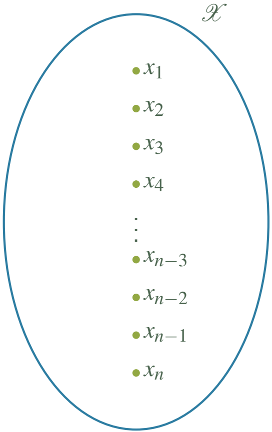
HOMEWORK EXERCISE A: Show that $u \gtrdot 0 \Rightarrow u \succ 0 \Rightarrow u > 0$.

HOMEWORK EXERCISE B: Show that $\mathrm{Inf}_{\gtrdot} = \mathrm{Inf}_{\succ} = \mathrm{Inf}_{>}$.


IP ON
VECTOR
SPACES
CLAS-
SICAL
QUAN-
TUM
lower
previsions
desirability

CLASSICAL PROBABILITY

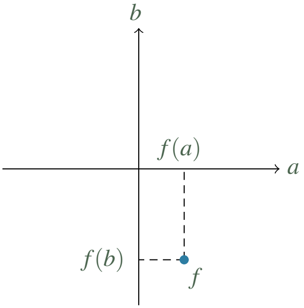
Classical probability: gambles as options



Consider a variable X that assumes values in some non-empty set $\mathscr{X}$, but whose value You don't know.

Classical probability: gambles as options

$\mathscr{X} = \{a, b\}$

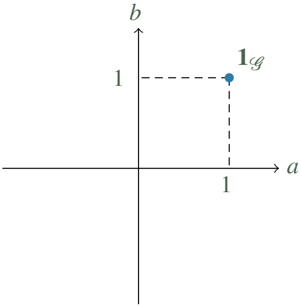


Consider a variable X that assumes values in some non-empty set $\mathscr{X}$, but whose value You don't know.

A bounded map $f\colon \mathscr{X} \to \mathbb{R}$ represents an uncertain reward $f(X)$; we call it a gamble.

Classical probability: gambles as options

$\mathscr{X} = \{a, b\}$



Consider a variable X that assumes values in some non-empty set $\mathscr{X}$, but whose value You don't know.

A bounded map $f\colon \mathscr{X} \to \mathbb{R}$ represents an uncertain reward $f(X)$; we call it a gamble.

The gambles constitute a normed linear space $\mathscr{G}$ with norm

$$\|f\|_\infty := \sup_{x \in \mathscr{X}} |f(x)| = \sup |f|$$

and as unit gamble $\mathbf{1}_{\mathscr{G}}$ the constant map 1:

$$\mathbf{1}_{\mathscr{G}}(x) = 1 \text{ for all } x \in \mathscr{X}.$$

Classical probability: background ordering

The background ordering $\succ$

is the strict vector ordering on the vector space $\mathscr{G}$ that is always there, regardless of what You may believe or prefer:

it represents COMPLETE IGNORANCE.

So, what is Your set of desirable gambles D under complete ignorance?

Classical probability: background ordering

The background ordering $\succ$

is the strict vector ordering on the vector space $\mathscr{G}$ that is always there, regardless of what You may believe or prefer:

it represents COMPLETE IGNORANCE.

So, what is Your set of desirable gambles D under complete ignorance?

$A :=$ Your set of accepted gambles

$R :=$ Your set of rejected gambles

$D :=$ Your set of desirable gambles $= A \cap -R$

$I :=$ Your set of indifferent gambles $= A \cap -A$

NO CONFUSION: $A \cap R = \emptyset$.

DESIRABILITY FRAMEWORK: $0 \in A$ and $A = D \cup I$ and $R = -D$.

International Journal of Approximate Reasoning 57 (2015) 69–102

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Accept & reject statement-based uncertainty models

Erik Quaeghebeur[a,b,*,1], Gert de Cooman[a], Filip Hermans[a]

[a] SYSTeMS Research Group, Ghent University, Technologiepark 914, 9052 Zwijnaarde, Belgium

[b] Centrum Wiskunde & Informatica, Postbus 94079, 1090 GB Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Received 11 July 2013
Received in revised form 5 December 2014
Accepted 16 December 2014
Available online 24 December 2014

Keywords:
Acceptability
Indifference
Desirability
Favourability
Preference
Prevision

ABSTRACT

We develop a framework for modelling and reasoning with uncertainty based on accept and reject statements about gambles. It generalises the frameworks found in the literature based on statements of acceptability, desirability, or favourability and clarifies their relative position. Next to the statement-based formulation, we also provide a translation in terms of preference relations, discuss—as a bridge to existing frameworks—a number of simplified variants, and show the relationship with prevision-based uncertainty models. We furthermore provide an application to modelling symmetry judgements.

Classical probability: background ordering

The one with constants:

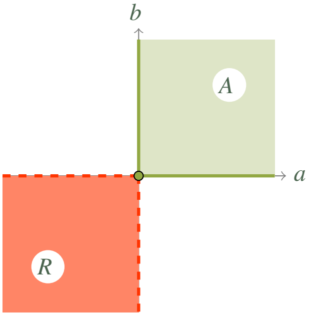
$$\frac{\mu \geq 0 \Rightarrow \mu \mathbf{1}_{\mathscr{G}} \in A \qquad \left(\inf f \geq \mu \text{ and } \mu \mathbf{1}_{\mathscr{G}} \in A\right) \Rightarrow f \in A}{\inf f \geq 0 \Rightarrow f \in A}$$

$$\frac{\mu < 0 \Rightarrow \mu \mathbf{1}_{\mathscr{G}} \in R \qquad \left(\sup f \leq \mu \text{ and } \mu \mathbf{1}_{\mathscr{G}} \in R\right) \Rightarrow f \in R}{\sup f < 0 \Rightarrow f \in R.}$$

Classical probability: background ordering

The one with constants:

$\mathscr{X} = \{a, b\}$



$$\frac{\mu \geq 0 \Rightarrow \mu \mathbf{1}_{\mathscr{G}} \in A \qquad \big(\inf f \geq \mu \text{ and } \mu \mathbf{1}_{\mathscr{G}} \in A\big) \Rightarrow f \in A}{\inf f \geq 0 \Rightarrow f \in A}$$

$$\frac{\mu < 0 \Rightarrow \mu \mathbf{1}_{\mathscr{G}} \in R \qquad \big(\sup f \leq \mu \text{ and } \mu \mathbf{1}_{\mathscr{G}} \in R\big) \Rightarrow f \in R}{\sup f < 0 \Rightarrow f \in R.}$$

So:

$$\{f \in \mathscr{G} : \inf f \geq 0\} \subseteq A \text{ and } \{f \in \mathscr{G} : \sup f < 0\} \subseteq R \tag{1}$$

Classical probability: background ordering

The one with complete ignorance:

$$\big(f \in A \text{ and } g(\mathscr{X}) \subseteq f(\mathscr{X})\big) \Rightarrow g \in A$$

As a consequence:

$$\begin{aligned} f \in A &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \in A \text{ for all } x \in \mathscr{X} && \text{[Complete Ignorance]} \\ &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \notin R \text{ for all } x \in \mathscr{X} && \text{[No Confusion]} \\ &\Rightarrow f(x) \geq 0 \text{ for all } x \in \mathscr{X} && \text{[Eq. (1)]} \\ &\Rightarrow \inf f \geq 0 \end{aligned}$$

Classical probability: background ordering

The one with complete ignorance:

$$\big(f \in A \text{ and } g(\mathscr{X}) \subseteq f(\mathscr{X})\big) \Rightarrow g \in A$$

As a consequence:

$$\begin{aligned} f \in A &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \in A \text{ for all } x \in \mathscr{X} && \text{[Complete Ignorance]} \\ &\Rightarrow f(x)\mathbf{1}_{\mathscr{G}} \notin R \text{ for all } x \in \mathscr{X} && \text{[No Confusion]} \\ &\Rightarrow f(x) \geq 0 \text{ for all } x \in \mathscr{X} && \text{[Eq. (1)]} \\ &\Rightarrow \inf f \geq 0 \end{aligned}$$

So:

$$A \subseteq \{f \in \mathscr{G} :\ \inf f \geq 0\} \tag{2}$$

$$\{f \in \mathscr{G} :\ \inf f \geq 0\} \subseteq A$$

Classical probability: background ordering

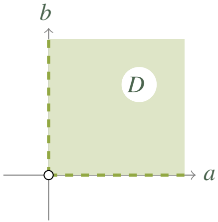
The one with the conclusion:

Combining Eqs. (1) and (2) with No Confusion yields:

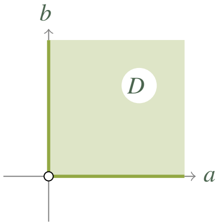
$$\frac{\begin{array}{r} A = \{f \in \mathscr{G} \colon \inf f \geq 0\} \\ \{f \in \mathscr{G} \colon \sup f < 0\} \subseteq R \subseteq \{f \in \mathscr{G} \colon \inf f < 0\} \end{array}}{\frac{\begin{array}{r} -A = \{f \in \mathscr{G} \colon \sup f \leq 0\} \\ \{f \in \mathscr{G} \colon \inf f > 0\} \subseteq -R \subseteq \{f \in \mathscr{G} \colon \sup f > 0\} \end{array}}{\begin{array}{c} \{f \in \mathscr{G} \colon \inf f > 0\} \subseteq D \subseteq \{f \in \mathscr{G} \colon \inf f \geq 0 \text{ and } f \neq 0\} \\ I = \{0\}. \end{array}}}$$

Classical probability: background ordering

strong dominance ordering



weak dominance ordering



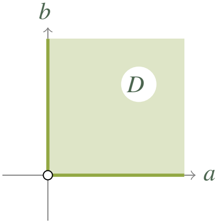
The one with the conclusion:

Combining Eqs. (1) and (2) with No Confusion yields:

$$\begin{array}{c}
A = \{f \in \mathscr{G} \colon \inf f \geq 0\} \\
\{f \in \mathscr{G} \colon \sup f < 0\} \subseteq R \subseteq \{f \in \mathscr{G} \colon \inf f < 0\} \\
\hline
-A = \{f \in \mathscr{G} \colon \sup f \leq 0\} \\
\{f \in \mathscr{G} \colon \inf f > 0\} \subseteq -R \subseteq \{f \in \mathscr{G} \colon \sup f > 0\} \\
\hline
\{f \in \mathscr{G} \colon \inf f > 0\} \subseteq D \subseteq \{f \in \mathscr{G} \colon \inf f \geq 0 \text{ and } f \neq 0\} \\
I = \{0\}.
\end{array}$$

Classical probability: background ordering

weak dominance ordering



The one with the conclusion:

Combining Eqs. (1) and (2) with No Confusion yields:

$$\frac{\begin{array}{r} A = \{f \in \mathscr{G} \colon \inf f \geq 0\} \\ \{f \in \mathscr{G} \colon \sup f < 0\} \subseteq R \subseteq \{f \in \mathscr{G} \colon \inf f < 0\} \end{array}}{\frac{\begin{array}{r} -A = \{f \in \mathscr{G} \colon \sup f \leq 0\} \\ \{f \in \mathscr{G} \colon \inf f > 0\} \subseteq -R \subseteq \{f \in \mathscr{G} \colon \sup f > 0\} \end{array}}{\begin{array}{c} \{f \in \mathscr{G} \colon \inf f > 0\} \subseteq D \subseteq \{f \in \mathscr{G} \colon \inf f \geq 0 \text{ and } f \neq 0\} \\ I = \{0\}. \end{array}}}$$

In the DESIRABILITY FRAMEWORK, the only remaining possibility is

$$D = \{f \in \mathscr{G} \colon \inf f \geq 0 \text{ and } f \neq 0\} \text{ and } I = \{0\},$$

so

$$f \succ 0 \Leftrightarrow \inf f \geq 0 \text{ and } f \neq 0.$$

Classical probability: background ordering

EXERCISE 1: What is vacuous lower prevision $\mathrm{Inf}_{\succ}$ that corresponds to this background ordering $\succ$?

Classical probability: background ordering

EXERCISE 1: What is vacuous lower prevision $\mathrm{Inf}_{\succ}$ that corresponds to this background ordering $\succ$?

$$\begin{aligned}
\mathrm{Inf}_{\succ} f &= \sup\{\alpha \in \mathbb{R} : f - \alpha \mathbf{1}_{\mathscr{G}} \succ 0\} \\
&= \sup\{\alpha \in \mathbb{R} :\ \inf(f - \alpha \mathbf{1}_{\mathscr{G}}) \geq 0 \text{ and } f \neq \alpha \mathbf{1}_{\mathscr{G}}\} \\
&= \sup\{\alpha \in \mathbb{R} :\ \inf f \geq \alpha \text{ and } f \neq \alpha \mathbf{1}_{\mathscr{G}}\} \\
&= \inf f.
\end{aligned}$$

and therefore the background ordering $\succ$ is also the weak dominance ordering $>$ of our more general context.

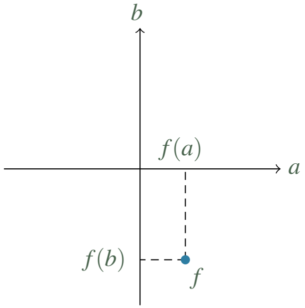
Classical probability: coherent previsions

The coherent previsions

Expansion in the standard basis:

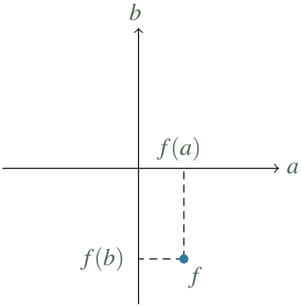
$\mathscr{X} = \{a, b\}$

$$f = \sum_{x \in \mathscr{X}} f(x)\mathbb{I}_{\{x\}}.$$



Classical probability: coherent previsions

$\mathscr{X} = \{a, b\}$



The coherent previsions

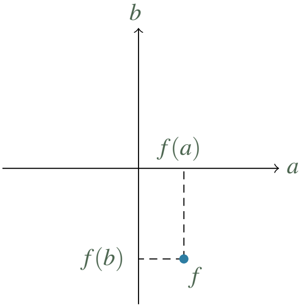
Expansion in the standard basis:

$$f = \sum_{x \in \mathscr{X}} f(x) \mathbb{I}_{\{x\}}.$$

EXERCISE 2: What is the prevision $P(f)$ of a gamble f?

Classical probability: coherent previsions

$\mathscr{X} = \{a, b\}$



The coherent previsions

Expansion in the standard basis:

$$f = \sum_{x \in \mathscr{X}} f(x) \mathbb{I}_{\{x\}}.$$

EXERCISE 2: What is the prevision $P(f)$ of a gamble f?

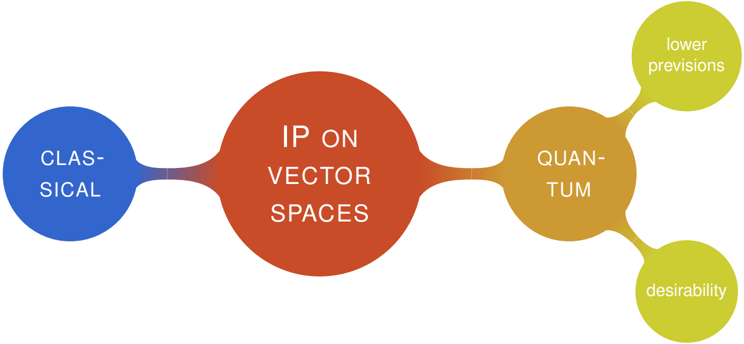
By linearity:

$$P(f) = \sum_{x \in \mathscr{X}} f(x) \underbrace{P(\mathbb{I}_{\{x\}})}_{p(x)} = E_p(f).$$

By the background condition:

$$\sum_{x \in \mathscr{X}} p(x) = 1 \text{ and } p(x) \geq 0.$$

P is a coherent prevision if and only if it's the expectation operator associated with a (finitely additive) probability measure.


CLAS-
SICAL
IP ON
VECTOR
SPACES
QUAN-
TUM
lower
previsions
desirability

QUANTUM PROBABILITY

Quantum IP: pioneers


ALESSIO BENAVOLI


ALESSANDRO FACCHINI


MARCO ZAFFALON


KEANO DE VOS


JASPER DE BOCK


ALEXANDER ERREYGERS


GERT DE COOMAN

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Kets and bras

The state of a quantum system is an element of a (here finite-dimensional) complex Hilbert space $\mathscr{X}$.

Elements of $\mathscr{X}$ are called state vectors or pure states or kets, and denoted by $|\phi\rangle$.

A Hilbert space has an inner product $(\bullet, \bullet)\colon \mathscr{X}^2 \to \mathbb{C}$.

The map

$$(|\psi\rangle, \bullet)\colon \mathscr{X} \to \mathbb{C}\colon |\phi\rangle \mapsto (|\psi\rangle, |\phi\rangle)$$

is a linear functional on $\mathscr{X}$, denoted by $\langle\psi|$, and called a dual state vector or bra.

The inner product $(|\psi\rangle, |\phi\rangle)$ of $|\phi\rangle$ and $|\psi\rangle$ is a complex number that results from the action of the bra $\langle\psi|$ on the ket $|\phi\rangle$, resulting in

$$\langle\psi|\phi\rangle := (|\psi\rangle, |\phi\rangle).$$

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Normal and orthogonal

If

$$\langle\phi|\phi\rangle = (|\phi\rangle, |\phi\rangle) = 1,$$

then $|\phi\rangle$ is normal(ised).

If

$$\langle\psi|\phi\rangle = (|\psi\rangle, |\phi\rangle) = 0,$$

then $|\phi\rangle$ and $|\psi\rangle$ are orthogonal.

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Linear operators and their adjoints

A linear operator $\hat{A}$ on $\mathscr{X}$ is a linear map $\hat{A}\colon \mathscr{X} \to \mathscr{X}$:

$$\hat{A}(\lambda|\phi\rangle + \mu|\psi\rangle) = \lambda\hat{A}|\phi\rangle + \mu\hat{A}|\psi\rangle \text{ for all } |\phi\rangle, |\psi\rangle \in \mathscr{X} \text{ and } \lambda, \mu \in \mathbb{C}.$$

The adjoint $\hat{A}^\dagger$ of $\hat{A}$ is the unique linear operator on $\mathscr{X}$ such that

$$\left(\hat{A}^\dagger|\psi\rangle, |\phi\rangle\right) = \left(|\psi\rangle, \hat{A}|\phi\rangle\right) \text{ for all } |\phi\rangle, |\psi\rangle \in \mathscr{X}.$$

$$\left(|\psi\rangle, \hat{A}|\phi\rangle\right) = \left(\hat{A}^\dagger|\psi\rangle, |\phi\rangle\right) = \overline{\left(|\phi\rangle, \hat{A}^\dagger|\psi\rangle\right)} = \overline{\left((\hat{A}^\dagger)^\dagger|\phi\rangle, |\psi\rangle\right)}$$
$$= \left(|\psi\rangle, (\hat{A}^\dagger)^\dagger|\phi\rangle\right)$$

so

$$(\hat{A}^\dagger)^\dagger = \hat{A}.$$

Quantum probability: mathematical background

Hermitian operators

A linear operator $\hat{A}$ is self-adjoint or Hermitian if $\hat{A} = \hat{A}^\dagger$, so

$$\begin{aligned}(\hat{A}|\psi\rangle, |\phi\rangle) &= (|\psi\rangle, \hat{A}|\phi\rangle) \\ &=: \langle\psi|\hat{A}|\phi\rangle \text{ for all } |\phi\rangle, |\psi\rangle \in \mathscr{X}.\end{aligned}$$

Quadratic forms of Hermitian operators are real:

$$\langle\phi|\hat{A}|\phi\rangle = (|\phi\rangle, \hat{A}|\phi\rangle) = (\hat{A}|\phi\rangle, |\phi\rangle) = \overline{(|\phi\rangle, \hat{A}|\phi\rangle)} = \overline{\langle\phi|\hat{A}|\phi\rangle} \in \mathbb{R}$$

The Hermitian operators on $\mathscr{X}$ constitute a real vector space $\mathscr{H}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

$$\dim(\mathscr{H}) = \dim(\mathscr{X})^2.$$

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Linear operators and their eigenkets and eigenvalues

If

$$\hat{A}|\phi\rangle = \lambda|\phi\rangle,$$

then $|\phi\rangle$ is an eigenket of $\hat{A}$ with eigenvalue λ.

All the eigenvectors that correspond to the same eigenvalue λ form a subspace $\mathscr{E}_\lambda$ of $\mathscr{X}$, called the eigenspace of λ.

If $\hat{A}$ is Hermitian, then its eigenvalues are real and its eigenspaces are orthogonal:

$$\lambda_1 \neq \lambda_2 \Rightarrow \langle\psi|\phi\rangle = 0 \text{ for all } |\phi\rangle \in \mathscr{E}_{\lambda_1} \text{ and } |\psi\rangle \in \mathscr{E}_{\lambda_2}.$$

We can always find an orthonormal basis of $\mathscr{X}$ consisting of eigenvectors of a Hermitian operator.

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Projection operators

A projection operator $\hat{P}$ is a Hermitian operator whose only eigenvalues are 0 and 1, so

$$\hat{P}|\phi\rangle = |\phi\rangle \text{ or } \hat{P}|\phi\rangle = 0,$$

and therefore $\hat{P}^2 = \hat{P}$, and

$$\mathscr{E}_0 = \ker \hat{P} \text{ and } \mathscr{E}_1 = \hat{P}\mathscr{X} = \{\hat{P}|\phi\rangle : |\phi\rangle \in \mathscr{X}\} \text{ and } \mathscr{E}_1 \oplus \mathscr{E}_0 = \mathscr{X}.$$

For every normalised $|\phi\rangle \in \mathscr{X}$, the linear operator

$$|\phi\rangle\langle\phi| : |\psi\rangle \mapsto (|\phi\rangle\langle\phi|)|\psi\rangle = |\phi\rangle\langle\phi|\psi\rangle = \langle\phi|\psi\rangle|\phi\rangle$$

is a projection operator that projects orthogonally onto the one-dimensional subspace spanned by $|\phi\rangle$.

Quantum probability: mathematical background

Projection operators

For any $|\phi\rangle \in \mathscr{X}$:

$$|\phi\rangle = \lambda|\psi_1\rangle + \mu|\psi_0\rangle \text{ with } \langle\psi_1|\psi_0\rangle = 0$$

and then

$$\begin{aligned}\langle\phi|\hat{P}|\phi\rangle &= (\overline{\lambda}\langle\psi_1| + \overline{\mu}\langle\psi_0|)\hat{P}(\lambda|\psi_1\rangle + \mu|\psi_0\rangle) \\ &= (\overline{\lambda}\langle\psi_1| + \overline{\mu}\langle\psi_0|)\lambda|\psi_1\rangle \\ &= |\lambda|^2\langle\psi_1|\psi_1\rangle \geq 0.\end{aligned}$$

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Hermitian operators and projections

If $\hat{A}$ is a Hermitian operator, then

$$\hat{A} = \sum_{\lambda \in \mathrm{spec}(\hat{A})} \lambda \hat{P}_{\mathscr{E}}$$

and then

$$\langle\phi|\hat{A}|\phi\rangle = \sum_{\lambda \in \mathrm{spec}(\hat{A})} \lambda \underbrace{\langle\phi|\hat{P}_{\mathscr{E}}|\phi\rangle}_{\geq 0}$$

so $\langle\bullet|\hat{A}|\bullet\rangle \geq 0$ everywhere if and only if

$$\min \mathrm{spec}(\hat{A}) \geq 0$$

which means that

A is positive semidefinite.

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Density operators

For any $p_1, \ldots, p_n \in \mathbb{R}_{>0}$ such that $\sum_{k=1}^n p_k = 1$, and any linearly independent $|\phi_1\rangle, \ldots, |\phi_n\rangle$ in $\mathscr{X}$, the linear operator

$$\hat{\rho} := \sum_{k=1}^{n} p_k |\phi_k\rangle\langle\phi_k|$$

is a density operator.

Observe that

– $\hat{\rho}$ is Hermitian

– $\min \operatorname{spec}(\hat{\rho}) \geq 0$, so $\hat{\rho} \geq 0$ is positive semidefinite

– $\operatorname{Tr}(\hat{\rho}) = \sum_{k=1}^n p_k = 1$

Quantum probability: mathematical background

Properties of an inner product:

(i) $(|\phi\rangle, |\phi\rangle) \geq 0$;

(ii) $(|\phi\rangle, |\phi\rangle) = 0 \Leftrightarrow |\phi\rangle = 0$;

(iii) $(|\phi\rangle, |\psi\rangle) = \overline{(|\psi\rangle, |\phi\rangle)}$;

(iv) $(\bullet, \lambda|\phi\rangle + \mu|\psi\rangle) = \lambda(\bullet, |\phi\rangle) + \mu(\bullet, |\psi\rangle)$.

Measurement in quantum mechanics

Any quantum system is in some normalised state $|\phi\rangle \in \mathscr{X}$ with $\langle\phi|\phi\rangle$=1.

Any measurement on the system

- corresponds to a Hermitian operator $\hat{A}$, and
- the possible outcomes are the eigenvalues $\lambda \in \mathrm{spec}(\hat{A})$.

Quantum probability: measurements as options

Consider a quantum system whose unknown state $|\Psi\rangle$ lives in an n-dimensional complex Hilbert state space $\mathscr{X}$, so

$$|\Psi\rangle \in \mathscr{X} \text{ and } \langle\Psi|\Psi\rangle = 1.$$

Any measurement corresponds to some Hermitian operator $\hat{A}$, and its outcome is uncertain. Its reward is the measurement outcome.

All measurements constitute a real n^2-dimensional Hilbert space $\mathscr{H}$, with (Frobenius) inner product

$$(\hat{A},\hat{B}) := \mathrm{Tr}(\hat{A}^\dagger\hat{B}) = \mathrm{Tr}(\hat{A}\hat{B}), \text{ for all } \hat{A},\hat{B} \in \mathscr{H},$$

corresponding (Frobenius) norm

$$\|\hat{A}\|_{\mathscr{H}} := \sqrt{(\hat{A},\hat{A})} = \sqrt{\mathrm{Tr}(\hat{A}^\dagger\hat{A})} \text{ for all } \hat{A} \in \mathscr{H},$$

and unit measurement $\mathbf{1}_{\mathscr{H}} := \hat{I}$.

Quantum probability: background ordering

The background ordering $\succ$

is the strict vector ordering on $\mathscr{H}$ that is always there, regardless of what You may believe or prefer:

it represents COMPLETE IGNORANCE.

EXERCISE 3: What is Your set of desirable measurements D under complete ignorance?

Quantum probability: background ordering

The background ordering $\succ$

is the strict vector ordering on $\mathscr{H}$ that is always there, regardless of what You may believe or prefer:

it represents COMPLETE IGNORANCE.

EXERCISE 3: What is Your set of desirable measurements D under complete ignorance?

$A :=$ Your set of accepted measurements

$R :=$ Your set of rejected measurements

$D :=$ Your set of desirable measurements $= A \cap -R$

$I :=$ Your set of indifferent measurements $= A \cap -A$

NO CONFUSION: $A \cap R = \emptyset$.

DESIRABILITY FRAMEWORK: $0 \in A$ and $A = D \cup I$ and $R = -D$.

Quantum probability: background ordering

The one with constants:

$$\frac{\mu \geq 0 \Rightarrow \mu\hat{I} \in A \qquad \left(\min\operatorname{spec}(\hat{A}) \geq \mu \text{ and } \mu\hat{I} \in A\right) \Rightarrow \hat{A} \in A}{\min\operatorname{spec}(\hat{A}) \geq 0 \Rightarrow \hat{A} \in A}$$

$$\frac{\mu < 0 \Rightarrow \mu\hat{I} \in R \qquad \left(\max\operatorname{spec}(\hat{A}) \leq \mu \text{ and } \mu\hat{I} \in R\right) \Rightarrow \hat{A} \in R}{\max\operatorname{spec}(\hat{A}) < 0 \Rightarrow \hat{A} \in R.}$$

Quantum probability: background ordering

Recall that the Hermitian operator $\hat{A}$ is

positive semidefinite if

$$\min \operatorname{spec}(\hat{A}) \geq 0,$$

positive definite if

$$\min \operatorname{spec}(\hat{A}) > 0,$$

and negative definite if

$$\max \operatorname{spec}(\hat{A}) < 0.$$

The one with constants:

$$\frac{\mu \geq 0 \Rightarrow \mu\hat{I} \in A \qquad \left(\min \operatorname{spec}(\hat{A}) \geq \mu \text{ and } \mu\hat{I} \in A\right) \Rightarrow \hat{A} \in A}{\min \operatorname{spec}(\hat{A}) \geq 0 \Rightarrow \hat{A} \in A}$$

$$\frac{\mu < 0 \Rightarrow \mu\hat{I} \in R \qquad \left(\max \operatorname{spec}(\hat{A}) \leq \mu \text{ and } \mu\hat{I} \in R\right) \Rightarrow \hat{A} \in R}{\max \operatorname{spec}(\hat{A}) < 0 \Rightarrow \hat{A} \in R.}$$

So:

$$\{\hat{A} \in \mathscr{H} : \ \min \operatorname{spec}(\hat{A}) \geq 0\} \subseteq A \text{ and } \{\hat{A} \in \mathscr{H} : \ \max \operatorname{spec}(\hat{A}) < 0\} \subseteq R \tag{3}$$

Quantum probability: background ordering

Recall that the Hermitian operator $\hat{A}$ is

positive semidefinite if

$$\min \operatorname{spec}(\hat{A}) \geq 0,$$

positive definite if

$$\min \operatorname{spec}(\hat{A}) > 0,$$

and negative definite if

$$\max \operatorname{spec}(\hat{A}) < 0.$$

The one with complete ignorance:

$$\left(\hat{A} \in A \text{ and } \operatorname{spec}(\hat{B}) \subseteq \operatorname{spec}(\hat{A})\right) \Rightarrow \hat{B} \in A$$

As a consequence:

$$\begin{aligned}
\hat{A} \in A &\Rightarrow \lambda \hat{I} \in A \text{ for all } \lambda \in \operatorname{spec}(\hat{A}) && \text{[Complete Ignorance]} \\
&\Rightarrow \lambda \hat{I} \notin R \text{ for all } \lambda \in \operatorname{spec}(\hat{A}) && \text{[No Confusion]} \\
&\Rightarrow \lambda \geq 0 \text{ for all } \lambda \in \operatorname{spec}(\hat{A}) && \text{[Eq. (3)]} \\
&\Rightarrow \min \operatorname{spec}(\hat{A}) \geq 0
\end{aligned}$$

Quantum probability: background ordering

Recall that the Hermitian operator $\hat{A}$ is

positive semidefinite if

$$\min \operatorname{spec}(\hat{A}) \geq 0,$$

positive definite if

$$\min \operatorname{spec}(\hat{A}) > 0,$$

and negative definite if

$$\max \operatorname{spec}(\hat{A}) < 0.$$

The one with complete ignorance:

$$\left(\hat{A} \in A \text{ and } \operatorname{spec}(\hat{B}) \subseteq \operatorname{spec}(\hat{A})\right) \Rightarrow \hat{B} \in A$$

As a consequence:

$$\begin{aligned} \hat{A} \in A &\Rightarrow \lambda \hat{I} \in A \text{ for all } \lambda \in \operatorname{spec}(\hat{A}) && \text{[Complete Ignorance]} \\ &\Rightarrow \lambda \hat{I} \notin R \text{ for all } \lambda \in \operatorname{spec}(\hat{A}) && \text{[No Confusion]} \\ &\Rightarrow \lambda \geq 0 \text{ for all } \lambda \in \operatorname{spec}(\hat{A}) && \text{[Eq. (3)]} \\ &\Rightarrow \min \operatorname{spec}(\hat{A}) \geq 0 \end{aligned}$$

So:

$$A \subseteq \{\hat{A} \in \mathscr{H} : \ \min \operatorname{spec}(\hat{A}) \geq 0\} \quad (4)$$

$$\{\hat{A} \in \mathscr{H} : \ \min \operatorname{spec}(\hat{A}) \geq 0\} \subseteq A$$

Quantum probability: background ordering

Recall that the Hermitian operator $\hat{A}$ is

positive semidefinite if

$$\min \operatorname{spec}(\hat{A}) \geq 0,$$

positive definite if

$$\min \operatorname{spec}(\hat{A}) > 0,$$

and negative definite if

$$\max \operatorname{spec}(\hat{A}) < 0.$$

The one with the conclusion:

Combining Eqs. (3) and (4) with No Confusion yields:

$$\frac{\dfrac{\begin{array}{c} A = \{\hat{A}:\ \min \operatorname{spec}(\hat{A}) \geq 0\} \\ \{\hat{A}:\ \max \operatorname{spec}(\hat{A}) < 0\} \subseteq R \subseteq \{\hat{A}:\ \min \operatorname{spec}(\hat{A}) < 0\} \end{array}}{\begin{array}{c} -A = \{\hat{A}:\ \max \operatorname{spec}(\hat{A}) \leq 0\} \\ \{\hat{A}:\ \min \operatorname{spec}(\hat{A}) > 0\} \subseteq -R \subseteq \{\hat{A}:\ \max \operatorname{spec}(\hat{A}) > 0\} \end{array}}}{\begin{array}{c} \{\hat{A}:\ \min \operatorname{spec}(\hat{A}) > 0\} \subseteq D \subseteq \{\hat{A}:\ \min \operatorname{spec}(\hat{A}) \geq 0 \text{ and } \hat{A} \neq 0\} \\ I = \{0\}. \end{array}}$$

Quantum probability: background ordering

Recall that the Hermitian operator $\hat{A}$ is

positive semidefinite if

$$\min \operatorname{spec}(\hat{A}) \geq 0,$$

positive definite if

$$\min \operatorname{spec}(\hat{A}) > 0,$$

and negative definite if

$$\max \operatorname{spec}(\hat{A}) < 0.$$

The one with the conclusion:

Combining Eqs. (3) and (4) with No Confusion yields:

$$\begin{array}{c} A = \{\hat{A} \colon \min \operatorname{spec}(\hat{A}) \geq 0\} \\ \{\hat{A} \colon \max \operatorname{spec}(\hat{A}) < 0\} \subseteq R \subseteq \{\hat{A} \colon \min \operatorname{spec}(\hat{A}) < 0\} \\ \hline -A = \{\hat{A} \colon \max \operatorname{spec}(\hat{A}) \leq 0\} \\ \{\hat{A} \colon \min \operatorname{spec}(\hat{A}) > 0\} \subseteq -R \subseteq \{\hat{A} \colon \max \operatorname{spec}(\hat{A}) > 0\} \\ \hline \{\hat{A} \colon \min \operatorname{spec}(\hat{A}) > 0\} \subseteq D \subseteq \{\hat{A} \colon \min \operatorname{spec}(\hat{A}) \geq 0 \text{ and } \hat{A} \neq 0\} \\ I = \{0\}. \end{array}$$

In the DESIRABILITY FRAMEWORK, the only remaining possibility is

$$D = \{\hat{A} \in \mathscr{H} \colon \min \operatorname{spec}(\hat{A}) \geq 0 \text{ and } \hat{A} \neq 0\} \text{ and } I = \{0\},$$

so

$$\hat{A} \succ 0 \Leftrightarrow \min \operatorname{spec}(\hat{A}) \geq 0 \text{ and } \hat{A} \neq 0.$$

Quantum probability: background ordering

EXERCISE 4: What is vacuous lower prevision $\mathrm{Inf}_{\succ}$ that corresponds to this background ordering $\succ$?

Quantum probability: background ordering

EXERCISE 4: What is vacuous lower prevision $\mathrm{Inf}_{\succ}$ that corresponds to this background ordering $\succ$?

$$
\begin{aligned}
\mathrm{Inf}_{\succ}\hat{A} &= \sup\{\alpha \in \mathbb{R} \colon \hat{A} - \alpha\mathbf{1}_{\mathscr{H}} \succ 0\} \\
&= \sup\{\alpha \in \mathbb{R} \colon\ \min\operatorname{spec}(\hat{A} - \alpha\hat{I}) \geq 0 \text{ and } \hat{A} \neq \alpha\hat{I}\} \\
&= \sup\{\alpha \in \mathbb{R} \colon\ \min\operatorname{spec}(\hat{A}) \geq \alpha \text{ and } \hat{A} \neq \alpha\hat{I}\} \\
&= \min\operatorname{spec}(\hat{A}).
\end{aligned}
$$

and therefore the background ordering $\succ$ is also the weak dominance ordering $>$ of our more general context.

Quantum probability: coherent previsions

The coherent previsions on $\mathscr{H}$

are the real functionals $P\colon \mathscr{H} \to \mathbb{R}$ that are:

- linear in the sense that

$$P(\lambda\hat{A}+\mu\hat{B}) = \lambda P(\hat{A}) + \mu P(\hat{B}) \text{ for all } \hat{A},\hat{B} \in \mathscr{H} \text{ and } \lambda,\mu \in \mathbb{R};$$

- bounded in the sense that

$$\min \operatorname{spec}(\hat{A}) \leq P(\hat{A}) \leq \max \operatorname{spec}(\hat{A}) \text{ for all } \hat{A} \in \mathscr{H}.$$

Quantum probability: coherent previsions

The one with linearity:

By the Riesz Representation Theorem, there is some unique $\hat{B}_P \in \mathscr{H}$ such that

$$P(\hat{A}) = (\hat{B}_P, \hat{A}) = \mathrm{Tr}(\hat{B}_P \hat{A}) \text{ for all } \hat{A} \in \mathscr{H}.$$

Quantum probability: coherent previsions

The one with linearity:

By the Riesz Representation Theorem, there is some unique $\hat{B}_P \in \mathscr{H}$ such that

$$P(\hat{A}) = (\hat{B}_P, \hat{A}) = \mathrm{Tr}(\hat{B}_P \hat{A}) \text{ for all } \hat{A} \in \mathscr{H}.$$

Recall that a density operator $\hat{\rho}$ is a positive semidefinite Hermitian operator such that $\mathrm{Tr}\,\hat{\rho} = 1$.

The one with boundedness:

EXERCISE 5: Show that $\hat{B}_P$ is a density operator.

Quantum probability: coherent previsions

Recall that a density operator $\hat{\rho}$ is a positive semidefinite Hermitian operator such that $\operatorname{Tr}\hat{\rho} = 1$.

The one with linearity:

By the Riesz Representation Theorem, there is some unique $\hat{B}_P \in \mathscr{H}$ such that

$$P(\hat{A}) = (\hat{B}_P, \hat{A}) = \operatorname{Tr}(\hat{B}_P \hat{A}) \text{ for all } \hat{A} \in \mathscr{H}.$$

The one with boundedness:

EXERCISE 5: Show that $\hat{B}_P$ is a density operator.

The one going backwards:

HOMEWORK EXERCISE C: Show that $\hat{A} \mapsto \operatorname{Tr}(\hat{\rho}\hat{A})$ is a coherent prevision for any density operator $\hat{\rho}$.

Quantum probability: coherent previsions

Conclusion: Born's rule

There is a one-to-one correspondence between coherent previsions P and density operators $\hat{\rho}$, with

$$P(\hat{A}) = \mathrm{Tr}(\hat{\rho}\hat{A}) \text{ for all } \hat{A} \in \mathscr{H}.$$

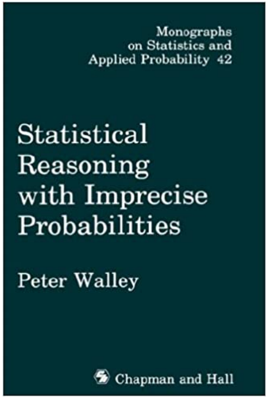
Recall that a density operator $\hat{\rho}$ is a positive semidefinite Hermitian operator such that $\mathrm{Tr}\,\hat{\rho} = 1$.

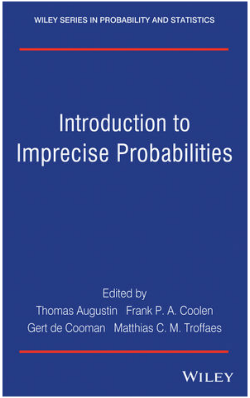
CONCLUSIONS

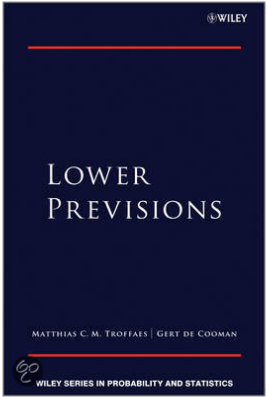
To conclude:

- Probability theory (in the form of desirability) is a deductive inference system, provided we allow for partial specification.
- probability = linearity + background ordering.
- the difference between classical and quantum probability lies only in the background ordering.
- the connection between classical and quantum probability is deep-rooted, and stares you in the face as soon as you have the right language to express probabilities in.
- what the background ordering is, is determined by what it means to be completely ignorant, and therefore also by the 'physics' of the problem.
- working with sets of desirable measurements = Heisenberg picture
- working with sets of density operators = Schrödinger picture

MORE RELEVANT LITERATURE


Monographs
on Statistics and
Applied Probability 42
Statistical
Reasoning
with Imprecise
Probabilities
Peter Walley
Chapman and Hall


WILEY SERIES IN PROBABILITY AND STATISTICS
Introduction to
Imprecise Probabilities
Edited by
Thomas Augustin Frank P. A. Coolen
Gert de Cooman Matthias C. M. Troffaes
WILEY


WILEY
LOWER
PREVISIONS
MATTHIAS C. M. TROFFAES | GERT DE COOMAN
WILEY SERIES IN PROBABILITY AND STATISTICS

PHYSICAL REVIEW A **94**, 042106 (2016)

Quantum mechanics: The Bayesian theory generalized to the space of Hermitian matrices

Alessio Benavoli,[*] Alessandro Facchini,[†] and Marco Zaffalon[‡]
Istituto Dalle Molle di Studi sull'Intelligenza Artificiale (IDSIA), Lugano, Switzerland
(Received 27 May 2016; published 10 October 2016)

We consider the problem of gambling on a quantum experiment and enforce rational behavior by a few rules. These rules yield, in the classical case, the Bayesian theory of probability via duality theorems. In our quantum setting, they yield the Bayesian theory generalized to the space of Hermitian matrices. This very theory is quantum mechanics: in fact, we derive all its four postulates from the generalized Bayesian theory. This implies that quantum mechanics is self-consistent. It also leads us to reinterpret the main operations in quantum mechanics as probability rules: Bayes' rule (measurement), marginalization (partial tracing), independence (tensor product). To say it with a slogan, we obtain that quantum mechanics is the Bayesian theory in the complex numbers.

DOI: 10.1103/PhysRevA.94.042106