RESEARCH PROJECT ON CONDITIONING IN QUANTUM PROBABILITY

GERT DE COOMAN

The context

In Thursday afternoon's lecture, we saw how both (conservative inference in) classical and quantum probability can be understood as special instances of a general theory of coherent sets of desirable options on abstract normed real vector spaces.

What we haven't touched upon yet, is the issue of conditioning. In this project, we'll investigate how conditioning can be understood in the context of quantum probability, and investigate whether the deep analogy between classical and quantum probability extends to conditioning as well.

Before getting to the quantum case, let's first see how conditioning fits into the desirability approach to *classical probability*.

The idea behind conditioning is that You start out with a coherent set of desirable gambles D that represents Your beliefs about the uncertain value of some variable X in a set $\mathscr{X}$. Then, You learn something new, namely that the variable X assumes value in the *non-empty* subset A of $\mathscr{X}$ *and nothing more*; we also say that You learn that *the event A has occurred*.

The question, then, is how to update Your coherent set of desirable gambles D to reflect this new information, and to obtain a new coherent set of desirable gambles $D|A$ that represents Your updated beliefs about the uncertain value of the variable X *given that the event A has occurred*.

The idea is to consider the *called-off* version $\mathbb{I}_A f$ of a gamble f, which is a new gamble that has the same values as the gamble f on the set A, and is zero—called off—elsewhere.

$$(\mathbb{I}_A f)(x) \coloneqq \mathbb{I}_A(x)f(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

So, in summary, we have the *indicator gamble* $\mathbb{I}_A$, which represents the event A at the gamble level and which assumes only *two possible values*, one on A and zero elsewhere; it's *combined with* the gamble f (using the pointwise multiplication operation) to obtain the called-off version $\mathbb{I}_A f$ of f:

$$(A, f) \mapsto \mathbb{I}_A f, \text{ or equivalently, } (\mathbb{I}_A, f) \mapsto \mathbb{I}_A f.$$

Now that we've come up with a way to combine (the indicator of) an event with a gamble to produce its called-off version, we use it in the following crucial argument. Since after learning that A has occurred, You know that the variable X assumes value in the set A, You should consider the gambles f and $\mathbb{I}_A f$ to be *equivalent*, as You know that they'll give You exactly the same reward. In fact, any gamble g such that $\mathbb{I}_A g = \mathbb{I}_A f$, or equivalently, that has the same values on A as f, should be considered *equivalent* to f, as You know that it will give You exactly the same reward as f does.

Now we let

$$D\|A \coloneqq \{f \in \mathscr{G} \colon \mathbb{I}_A f \in D\},$$

the set of all gambles that are desirable when called off on A. This set is *A-coarse*, in the sense that

$$(f \in D\|A \text{ and } \mathbb{I}_A f = \mathbb{I}_A g) \Rightarrow g \in D\|A, \text{ for all } f, g \in \mathscr{G}.$$

Date: SIPTA School 2024 – 16/8/2024.

This set $D\|A$ satisfies the coherence requirements D1, D2, and D3, but not necessarily D4. It would satisfy D4 if we also restricted the background ordering $>$ to A, to reflect that You now know that $X \in A$. In any case, this can also be remedied by explicitly adding the background gambles $\mathscr{G}_{>0}$ to $D\|A$:

$$D|A \coloneqq D\|A \cup \mathscr{G}_{>0}.$$

This set is coherent, and we call it the *conditional coherent set of desirable gambles* given that the event A has occurred. The associated price functional

$$\underline{\Lambda}_{D|A}(f) = \underline{\Lambda}_{D\|A}(f) = \sup\{\alpha \in \mathbb{R} \colon f - \alpha \mathbf{1}_{\mathscr{G}} \in D\|A\} = \sup\{\alpha \in \mathbb{R} \colon (f - \alpha \mathbf{1}_{\mathscr{G}})\mathbb{I}_A \in D\}$$

is the *conditional lower prevision* of the gamble f given that the event A has occurred.

Interestingly, when D is precise in the sense that there's some coherent prevision P such that $D = \{f \in \mathscr{G} \colon P(f) > 0\} \cup \mathscr{G}_{>0}$, then if $P(A) > 0$,

$$\begin{aligned}
\underline{\Lambda}_{D|A}(f) &= \sup\{\alpha \in \mathbb{R} \colon (f - \alpha \mathbf{1}_{\mathscr{G}})\mathbb{I}_A \in D\} \\
&= \sup\{\alpha \in \mathbb{R} \colon P((f - \alpha \mathbf{1}_{\mathscr{G}})\mathbb{I}_A) > 0\} \\
&= \sup\{\alpha \in \mathbb{R} \colon P(f\mathbb{I}_A) > \alpha P(A) > 0\} \\
&= \frac{P(f\mathbb{I}_A)}{P(A)},
\end{aligned}$$

so we have derived Bayes' rule for conditioning in classical probability.

The assignment

Can a similar argument be made for quantum probability? And in adapting the argument, what are the key differences between the classical and quantum cases? In particular,

- What is the quantum analogue (in terms of measurements) of an event and its indicator?
- How are these events combined with measurements to produce 'called-off measurements'?
- What is the quantum analogue of the set of all measurements that are desirable when called off on an event?
- What is the quantum analogue of the conditional coherent set of desirable measurements given an event?
- What is the quantum analogue of the conditional lower prevision of a measurement given an event?
- Can a quantum analogue of Bayes' rule for conditioning be derived? If so, what does it look like? Does what you've found have a name?