RESEARCH PROJECT ON IMPRECISE MARKOV CHAINS

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A (precise and time-homogeneous) Markov chain with a finite state space is (almost) completely defined by its *(transition operator* (or Markov kernel) $\mathrm{T}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$. In particular, for all $n \in \mathbb{N}$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$E(f(X_{n+1})) = E_{q_\square}(\mathrm{T}^n f).$$

An important type of transition operators are the *ergodic* ones, that is, those for which for all $f\colon \mathcal{X} \to \mathbb{R}$ and $x \in \mathcal{X}$, $\mathrm{T}^n f(x)$ converges to a limit that doesn't depend on x. There are many sufficient conditions for ergodicity, and one of the more important well-known ones is that it is *regular*, where a transition operator is regular if for all $x \in \mathcal{X}$, there is some $n \in \mathbb{N}$ such that $\min \mathrm{T}^n \mathbb{I}_x > 0$. Another way to define regularity is through the transition graph associated with T: the nodes of this graph are the elements of $\mathcal{X}$, and there is a directed edge from x to y if and only if $T(x, y) = \mathrm{T}\mathbb{I}_y(x) > 0$. For regularity, we now need that (i) this transition graph is strongly connected, meaning that there's a directed path from any state $x \in \mathcal{X}$ to any other state $y \in \mathcal{X}$; and (ii) for any state $x \in \mathcal{X}$, the greatest common divisor of the lengths of the directed paths that start and end in x is 1. A necessary condition for T to be ergodic, then, is that there is at least one state $x \in \mathcal{X}$ for which there is some $n_x \in \mathbb{N}$ such that for any state $y \in \mathcal{X}$ and $k \geq n_x$, there is a directed path of length k from y to x.

In the lecture of Thursday, we saw how imprecise Markov chains with a finite state space $\mathcal{X}$ are (almost) completely defined by an upper transition operator $\overline{\mathrm{T}}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$. Here too, we can define a notion of ergodicity: an upper transition operator $\overline{\mathrm{T}}$ is *ergodic* if for all $f \in \mathbb{R}^{\mathcal{X}}$ and $x \in \mathcal{X}$, $\overline{\mathrm{T}}^n f(x)$ converges to a limit that doesn't depend on x.

Of course, one can again come up with sufficient and necessary conditions for ergodicity. In fact, Hermans and Cooman [2] have found necessary and sufficient conditions similar to the ones for precise Markov chains mentioned above. The objective of this research project is to arrive at (some of) these necessary and sufficient conditions *without* looking at the papers [1–4].

References

[1] Gert de Cooman, Filip Hermans, and Erik Quaeghebeur. "Imprecise Markov Chains and their limit behavior". In: *Probability in the Engineering and Informational Sciences* 23.4 (Oct. 2009), pp. 597–635. DOI: 10.1017/S0269964809990039.

[2] Filip Hermans and Gert de Cooman. "Characterisation of ergodic upper transition operators". In: *International Journal of Approximate Reasoning* 53.4 (2012), pp. 573–583. DOI: 10.1016/j.ijar.2011.12.008.

[3] D. Škulj and R. Hable. "Coefficients of ergodicity for Markov chains with uncertain parameters". In: *Metrika* 76.1 (2013), pp. 107–133. DOI: 10.1007/s00184-011-0378-0.

[4] Natan T'Joens and Jasper De Bock. "Average behaviour in discrete-time imprecise Markov chains: A study of weak ergodicity". In: *International Journal of Approximate Reasoning* 132 (2021), pp. 181–205. DOI: 10.1016/j.ijar.2021.03.001.

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