Imprecise-probabilistic processes – Part I

Alexander Erreygers

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Uncertain process

We consider a system whose uncertain state X_t

takes values in the state space $\mathcal{X}$ and

changes over time $t \in \mathbb{T}$.


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Uncertain process

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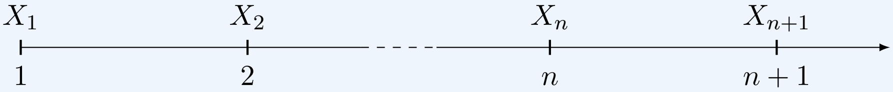
For now, let us focus on finite-state and discrete-time uncertain processes, so with

$$\mathcal{X} \text{ finite and } \mathbb{T} = \mathbb{N} = \{1, 2, 3, \dots\}.$$


X_1
X_2
X_n
X_{n+1}
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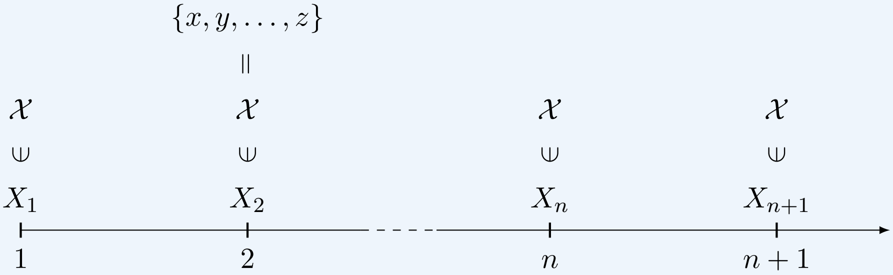
n
$n+1$


$\{x, y, \ldots, z\}$
$\|$
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X_1 X_2 X_n X_{n+1}
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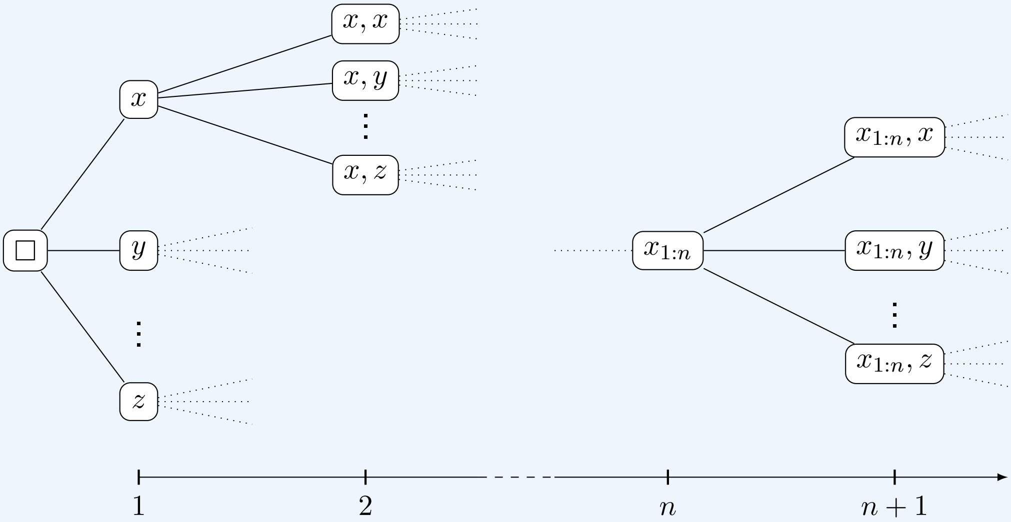

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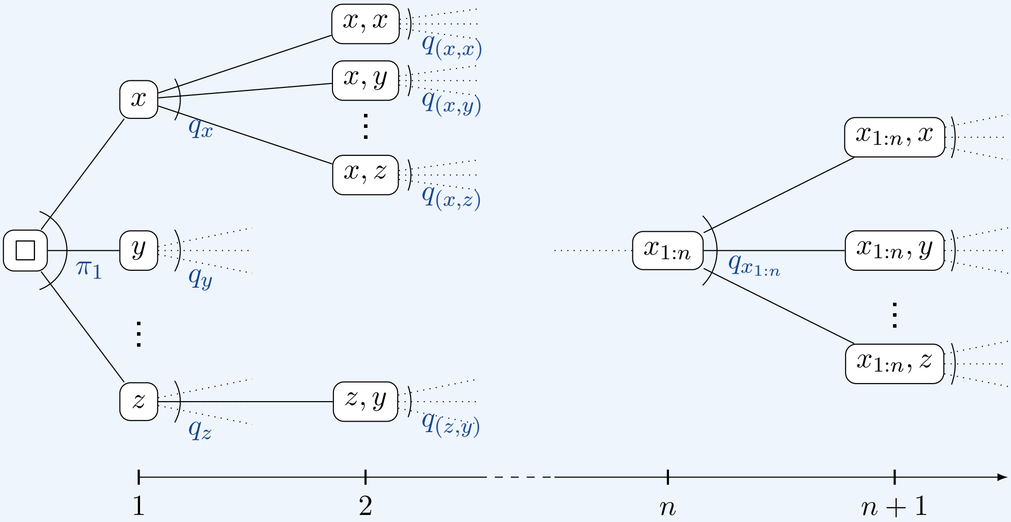

x, x
$q_{(x,x)}$
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$q_{(x,y)}$
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z, y
$q_{(z,y)}$
1
2
n
$n+1$

Precise probability tree

$$\pi_1 \in \Sigma_{\mathcal{X}}$$

$$(\forall n \in \mathbb{N}, x_{1:n} \in \mathcal{X}^n)\ q_{x_{1:n}} \in \Sigma_{\mathcal{X}}$$

$$P(X_1 = x) = \pi_1(x)$$

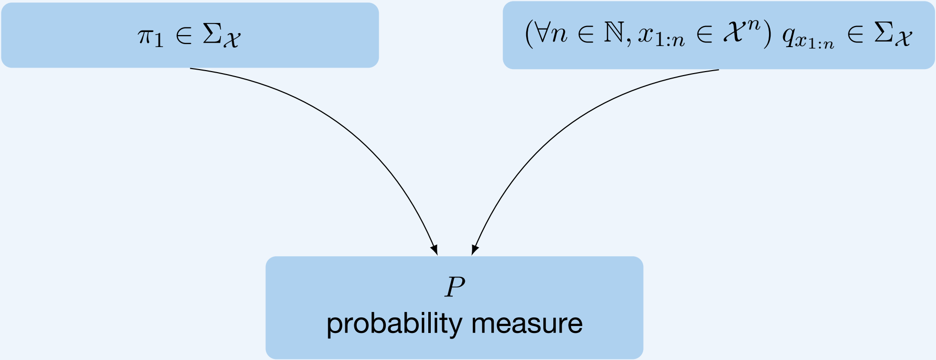
Precise probability tree

$$\pi_1 \in \Sigma_{\mathcal{X}}$$

$$(\forall n \in \mathbb{N}, x_{1:n} \in \mathcal{X}^n)\ q_{x_{1:n}} \in \Sigma_{\mathcal{X}}$$

$$P(X_{n+1} = x \mid X_1 = x_1, \ldots, X_n = x_n) = q_{x_{1:n}}(x)$$

Precise probability tree



From local to global probabilities

For any $n > 1$ and $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$P(X_{1:n} = x_{1:n}) = P(X_1 = x_1, \ldots, X_n = x_n).$$

From local to global probabilities

For any $n > 1$ and $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$
\begin{aligned}
P(X_{1:n} = x_{1:n}) &= P(X_1 = x_1, \ldots, X_n = x_n) \\
&= P(X_{1:n-1} = x_{1:n-1}) P(X_n = x_n \mid X_{1:n-1} = x_{1:n-1}).
\end{aligned}
$$

From local to global probabilities

For any $n > 1$ and $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$\begin{aligned} P(X_{1:n} = x_{1:n}) &= P(X_1 = x_1, \ldots, X_n = x_n) \\ &= P(X_{1:n-1} = x_{1:n-1}) P(X_n = x_n \mid X_{1:n-1} = x_{1:n-1}) \\ &= P(X_{1:n-1} = x_{1:n-1}) q_{x_{1:n-1}}(x_n). \end{aligned}$$

From local to global probabilities

For any $n > 1$ and $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$\begin{aligned}
P(X_{1:n} = x_{1:n}) &= P(X_1 = x_1, \ldots, X_n = x_n) \\
&= P(X_{1:n-1} = x_{1:n-1}) P(X_n = x_n \mid X_{1:n-1} = x_{1:n-1}) \\
&= P(X_{1:n-1} = x_{1:n-1}) q_{x_{1:n-1}}(x_n) \\
&= \cdots \\
&= P(X_1 = x_1) P(X_2 = x_2 \mid X_1 = x_1) q_{x_{1:2}}(x_3) \cdots q_{x_{1:n-1}}(x_n).
\end{aligned}$$

From local to global probabilities

For any $n > 1$ and $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$,

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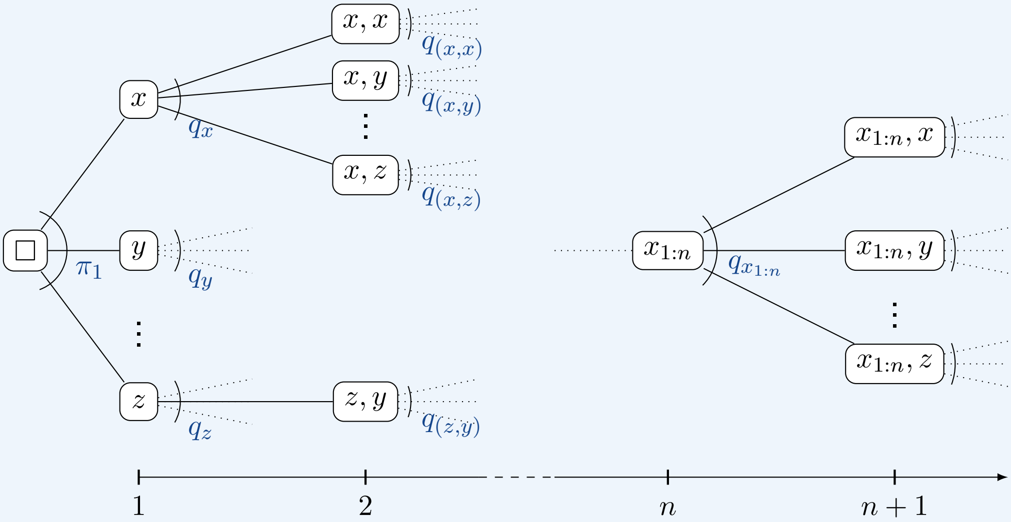
P(X_{1:n} = x_{1:n}) &= P(X_1 = x_1, \ldots, X_n = x_n) \\
&= P(X_{1:n-1} = x_{1:n-1}) P(X_n = x_n \mid X_{1:n-1} = x_{1:n-1}) \\
&= P(X_{1:n-1} = x_{1:n-1}) q_{x_{1:n-1}}(x_n) \\
&= \cdots \\
&= P(X_1 = x_1) P(X_2 = x_2 \mid X_1 = x_1) q_{x_{1:2}}(x_3) \cdots q_{x_{1:n-1}}(x_n) \\
&= \pi_1(x_1) q_{x_1}(x_2) \cdots q_{x_{1:n-1}}(x_n).
\end{aligned}$$


x, x
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$q_{(x,y)}$
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$x_{1:n}, x$
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q_y
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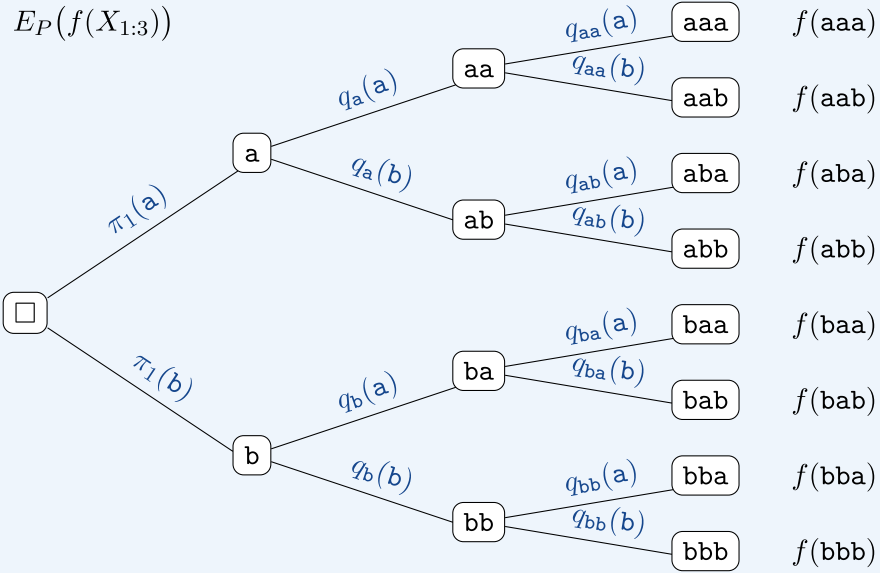
$q_{x_{1:n}}$
$x_{1:n}, y$
$x_{1:n}, z$
z
q_z
z, y
$q_{(z,y)}$
1
2
n
$n + 1$

From local probabilities to global expectations

For $n \in \mathbb{N}$ and $f\colon \mathcal{X}^n \to \mathbb{R}$,

$$\begin{aligned} E\big(f(X_{1:n})\big) &= \sum_{x_{1:n} \in \mathcal{X}^n} f(x_{1:n}) P(X_{1:n} = x_{1:n}) \\ &= \sum_{x_{1:n} \in \mathcal{X}^n} f(x_{1:n}) \pi_1(x_1) q_{x_1}(x_2) \cdots q_{x_{1:n-1}}(x_n). \end{aligned}$$

$E_P\big(f(X_{1:3})\big)$



From local probabilities to global expectations

For $n \in \mathbb{N}$ and $f \colon \mathcal{X}^n \to \mathbb{R}$,

$$
\begin{aligned}
E\big(f(X_{1:n})\big) &= \sum_{x_{1:n} \in \mathcal{X}^n} f(x_{1:n}) P(X_{1:n} = x_{1:n}) \\
&= \sum_{x_{1:n} \in \mathcal{X}^n} f(x_{1:n}) \pi_1(x_1) q_{x_1}(x_2) \cdots q_{x_{1:n-1}}(x_n). \\
&= \sum_{x_1 \in \mathcal{X}} \pi_1(x_1) \sum_{x_2 \in \mathcal{X}} q_{x_1}(x_2) \cdots \sum_{x_n \in \mathcal{X}} q_{x_{1:n-1}}(x_n) f(x_{1:n}).
\end{aligned}
$$

From local probabilities to global expectations

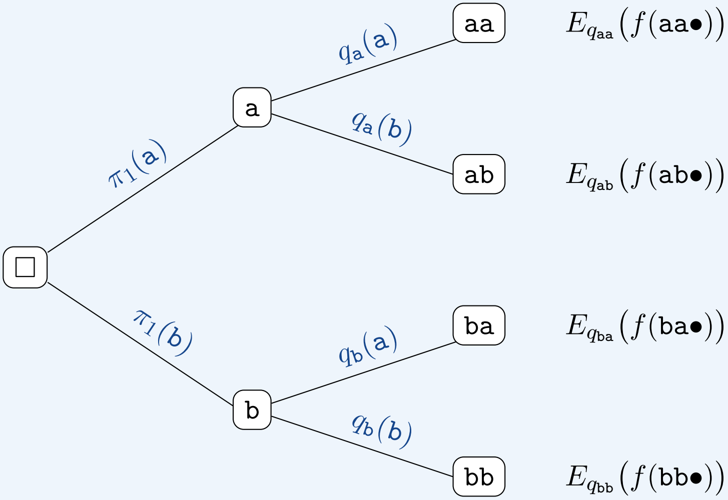
For $n \in \mathbb{N}$ and $f\colon \mathcal{X}^n \to \mathbb{R}$,

$$
\begin{aligned}
E\big(f(X_{1:n})\big) &= \sum_{x_{1:n}\in\mathcal{X}^n} f(x_{1:n}) P(X_{1:n} = x_{1:n}). \\
&= \sum_{x_{1:n}\in\mathcal{X}^n} f(x_{1:n}) \pi_1(x_1) q_{x_1}(x_2) \cdots q_{x_{1:n-1}}(x_n). \\
&= \sum_{x_1\in\mathcal{X}} \pi_1(x_1) \sum_{x_2\in\mathcal{X}} q_{x_1}(x_2) \cdots \sum_{x_n\in\mathcal{X}} q_{x_{1:n-1}}(x_n) f(x_{1:n}) \\
&= E\big(g(X_{1:n-1})\big)
\end{aligned}
$$

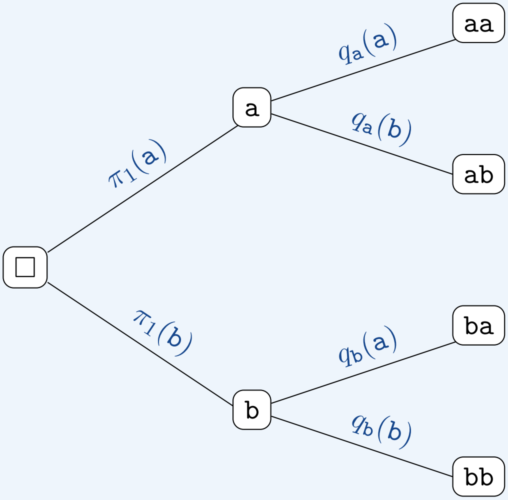
with

$$
g\colon \mathcal{X}^{n-1} \to \mathbb{R}\colon x_{1:n-1} \mapsto E_{q_{x_{1:n-1}}}\Big(\underbrace{f(x_{1:n-1}\bullet)}_{\mathcal{X}\to\mathbb{R}\colon x_n \mapsto f(x_{1:n})}\Big).
$$

$$E_P\big(f(X_{1:3})\big)$$



$$E_P\big(f(X_{1:3})\big) = E_P\big(g(X_{1:2})\big)$$



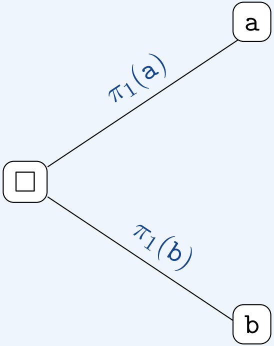
$$E_{q_{\mathtt{aa}}}\big(f(\mathtt{aa}\bullet)\big) =: g(\mathtt{aa})$$

$$E_{q_{\mathtt{ab}}}\big(f(\mathtt{ab}\bullet)\big) =: g(\mathtt{ab})$$

$$E_{q_{\mathtt{ba}}}\big(f(\mathtt{ba}\bullet)\big) =: g(\mathtt{ba})$$

$$E_{q_{\mathtt{bb}}}\big(f(\mathtt{bb}\bullet)\big) =: g(\mathtt{bb})$$

$$E_P\big(f(X_{1:3})\big) = E_P\big(h(X_1)\big)$$

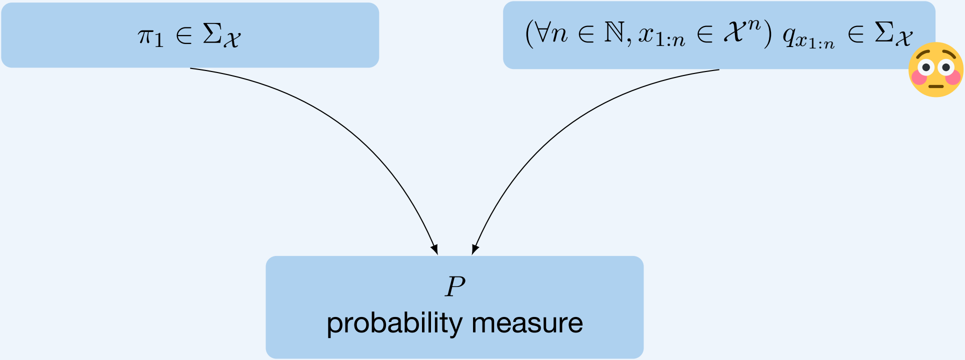


$$E_{q_{\mathtt{a}}}\big(g(\mathtt{a}\bullet)\big) =: h(\mathtt{a})$$

$$E_{q_{\mathtt{b}}}\big(g(\mathtt{b}\bullet)\big) =: h(\mathtt{b})$$

$$E_{\pi_1}(h) = E_P\big(f(X_{1:3})\big)$$

Precise probability tree



The Markov assumption

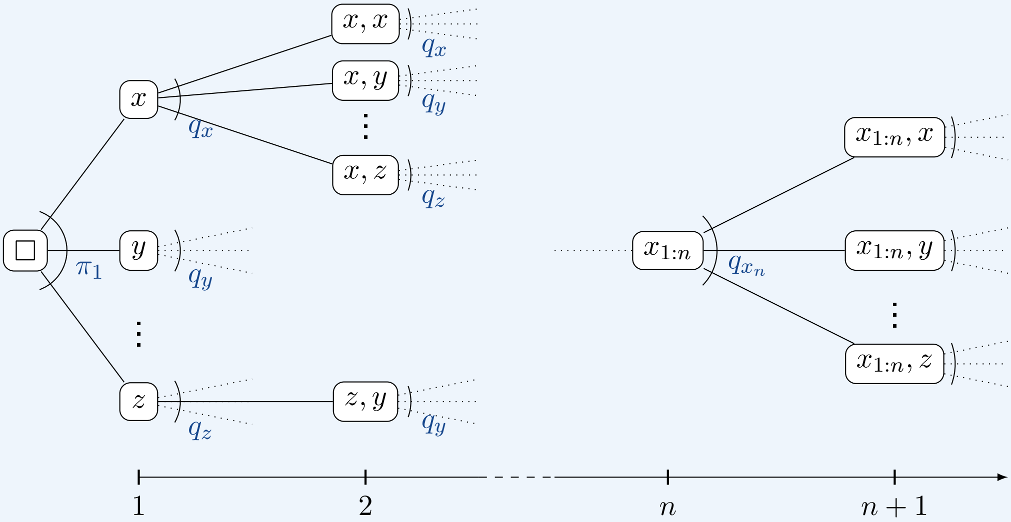
For all $n > 1$, $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ and $y \in \mathcal{X}$, we're going to assume that

$$\begin{aligned} &P(X_{n+1} = y \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x_n) \\ &\quad = P(X_{n+1} = y \mid X_n = x_n) \end{aligned}$$

The Markov and time-homogeneity assumptions

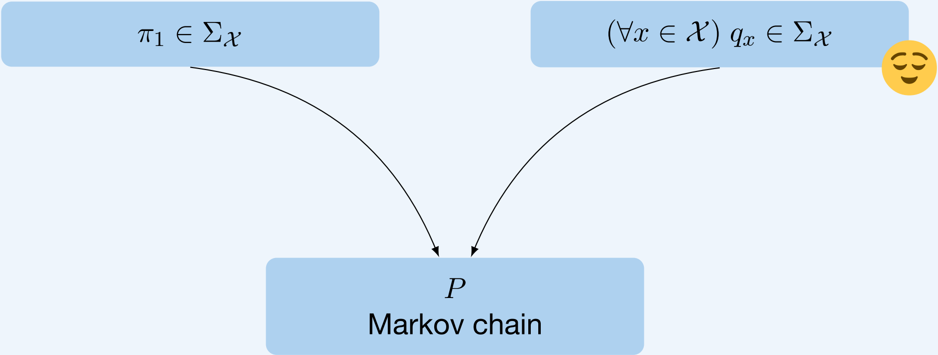
For all $n > 1$, $x_{1:n} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ and $y \in \mathcal{X}$, we're going to assume that

$$\begin{aligned} P(X_{n+1} = y \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x_n) \\ = P(X_{n+1} = y \mid X_n = x_n) \\ = P(X_2 = y \mid X_1 = x_n). \end{aligned}$$



$\square$
π_1
x
y
z
q_x
q_y
q_z
x, x
x, y
x, z
z, y
q_x
q_y
q_z
q_y
$x_{1:n}$
q_{x_n}
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
n
$n+1$

Markov chain



Markov chain

$\pi_1 \in \Sigma_{\mathcal{X}}$

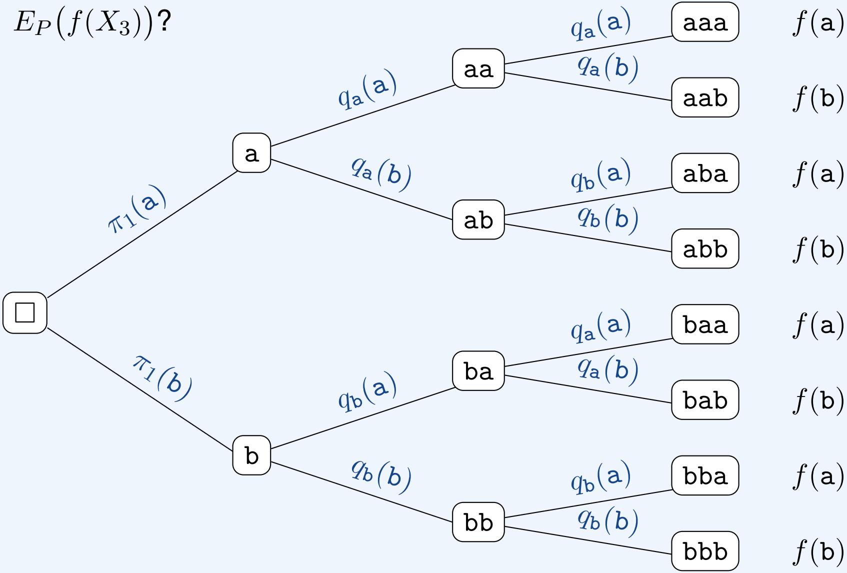
$(\forall x \in \mathcal{X})\ q_x \in \Sigma_{\mathcal{X}}$

$$P(X_{n+1} = y \mid X_{1:n} = x_{1:n}) = P(X_{n+1} = y \mid X_n = x_n) = P(X_2 = y \mid X_1 = x_n)$$

P

Markov chain

$E_P(f(X_3))$?


□
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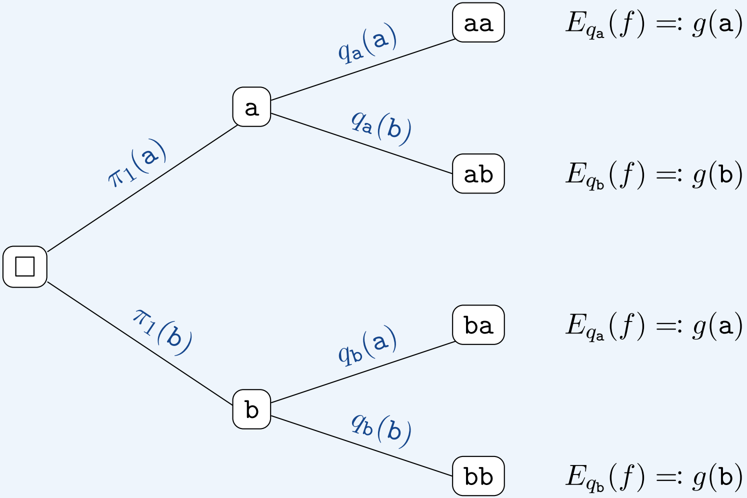
$f(a)$
$f(b)$
$f(a)$
$f(b)$
$f(a)$
$f(b)$
$f(a)$
$f(b)$

$$E_P\big(f(X_3)\big) = E(g(X_2))$$

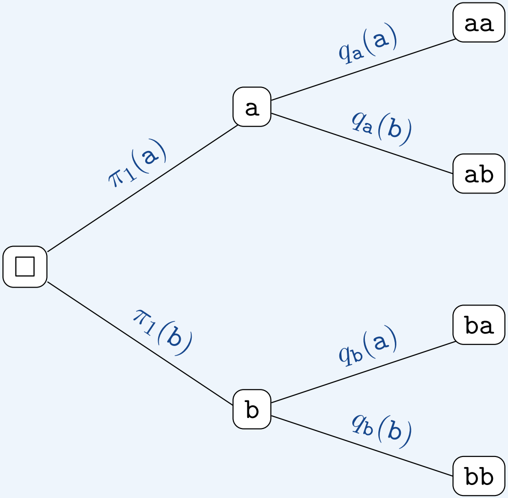


The transition operator

Let $\mathsf{T}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ be the corresponding *transition operator* which maps $f\colon \mathcal{X} \to \mathbb{R}$ to

$$\mathsf{T}f\colon \mathcal{X} \to \mathbb{R}\colon x \mapsto E_{q_x}(f) = \sum_{y\in\mathcal{X}} q_x(y)f(y).$$

$$E_P\big(f(X_3)\big) = E(\mathsf{T}f(X_2))$$



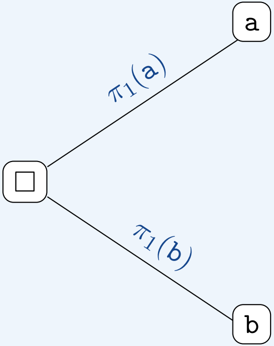
$$E_{q_{\mathsf{a}}}(f) = \mathsf{T}f(\mathsf{a})$$

$$E_{q_{\mathsf{b}}}(f) = \mathsf{T}f(\mathsf{b})$$

$$E_{q_{\mathsf{a}}}(f) = \mathsf{T}f(\mathsf{a})$$

$$E_{q_{\mathsf{b}}}(f) = \mathsf{T}f(\mathsf{b})$$

$$E_P\big(f(X_3)\big) = E(\mathsf{T}^2 f(X_1))$$



$$E_{q_{\mathtt{a}}}\big(\mathsf{T}f\big) = \mathsf{T}^2 f(\mathtt{a})$$

$$E_{q_{\mathtt{b}}}\big(\mathsf{T}f\big) = \mathsf{T}^2 f(\mathtt{b})$$

□

$$E_{\pi_1}(\mathsf{T}^2 f) = E_P\big(f(X_3)\big)$$

Global expectations via the transition operator

We've essentially shown that for all $n > 1$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$E\big(f(X_n)\big) = E_{\pi_1}(\mathsf{T}^{n-1} f).$$

Global expectations via the transition operator

We've essentially shown that for all $n > 1$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$E\big(f(X_n)\big) = E_{\pi_1}(\mathsf{T}^{n-1} f).$$

More generally, for $n > 1$ and $f\colon \mathcal{X}^n \to \mathbb{R}$,

$$E\big(f(X_{1:n})\big) = E\big(g(X_{1:n-1})\big) \qquad ,$$

with

$$g\colon \mathcal{X}^{n-1} \to \mathbb{R}\colon x_{1:n-1} \mapsto E_{q_{x_{n-1}}}\big(f(x_{1:n-1}\bullet)\big).$$

Global expectations via the transition operator

We've essentially shown that for all $n > 1$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$E\big(f(X_n)\big) = E_{\pi_1}(\mathsf{T}^{n-1} f)$$

More generally, for $n > 1$ and $f\colon \mathcal{X}^n \to \mathbb{R}$,

$$E\big(f(X_{1:n})\big) = E\big(g(X_{1:n-1})\big) = E\Big(\big[\mathsf{T} f(X_{1:n-1}\bullet)\big](X_{n-1})\Big),$$

since for all $x_{1:n-1} \in \mathcal{X}^{n-1}$,

$$g(x_{1:n-1}) = E_{q_{x_{n-1}}}\big(f(x_{1:n-1}\bullet)\big) = \big[\mathsf{T} f(x_{1:n-1}\bullet)\big](x_{n-1}).$$

The transition operator

Let $\mathsf{T}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ be the corresponding *transition operator* which maps $f\colon \mathcal{X} \to \mathbb{R}$ to

$$\mathsf{T}f\colon \mathcal{X} \to \mathbb{R}\colon x \mapsto E_{q_x}(f) = \sum_{y \in \mathcal{X}} q_x(y) f(y).$$

For some ordering $x_1, \ldots, x_m$ of $\mathcal{X}$ and with basis $\mathcal{B} = \{\mathbb{I}_{x_1}, \ldots, \mathbb{I}_{x_m}\}$, the transition operator T has the matrix representation

$$T = [\mathsf{T}]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} q_{x_1}(x_1) & \cdots & q_{x_1}(x_m) \\ \vdots & \ddots & \vdots \\ q_{x_m}(x_1) & \cdots & q_{x_m}(x_m) \end{bmatrix}.$$

Global expectations via the transition operator

We've essentially shown that for all $n > 1$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$E\big(f(X_n)\big) = E_{\pi_1}(\mathsf{T}^{n-1} f) = [\pi_1]_{\mathcal{B}}^\top \, T^{n-1} \, [f]_{\mathcal{B}},$$

with

$$[\pi_1]_{\mathcal{B}}^\top = \begin{bmatrix} \pi_1(x_1) & \cdots & \pi_1(x_m) \end{bmatrix} \quad \text{and} \quad [f]_{\mathcal{B}} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_m) \end{bmatrix}.$$

Exercise 1

For $\mathcal{X} = \{\mathtt{a}, \mathtt{b}\}$, $\pi_1 = \mathbb{I}_{\mathtt{a}}$, $[q_{\mathtt{a}}]_{\mathcal{B}} = \begin{bmatrix}\alpha & 1-\alpha\end{bmatrix}^\top$ and $[q_{\mathtt{b}}]_{\mathcal{B}} = \begin{bmatrix}1-\beta & \beta\end{bmatrix}^\top$, determine

$$P(X_3 = \mathtt{b}) = E(\mathbb{I}_{\mathtt{b}}(X_3)).$$

Exercise 1

For $\mathcal{X} = \{\mathtt{a}, \mathtt{b}\}$, $\pi_1 = \mathbb{I}_{\mathtt{a}}$, $[q_{\mathtt{a}}]_{\mathcal{B}} = \begin{bmatrix}\alpha & 1-\alpha\end{bmatrix}^\top$ and $[q_{\mathtt{b}}]_{\mathcal{B}} = \begin{bmatrix}1-\beta & \beta\end{bmatrix}^\top$,

$$\begin{aligned} P(X_3 = \mathtt{b}) &= E(\mathbb{I}_{\mathtt{b}}(X_3)) \\ &= [\pi_1]_{\mathcal{B}}^\top T^2 [\mathbb{I}_b]_{\mathcal{B}} \end{aligned}$$

Exercise 1

For $\mathcal{X} = \{\texttt{a}, \texttt{b}\}$, $\pi_1 = \mathbb{I}_{\texttt{a}}$, $[q_{\texttt{a}}]_{\mathcal{B}} = \begin{bmatrix} \alpha & 1-\alpha \end{bmatrix}^\top$ and $[q_{\texttt{b}}]_{\mathcal{B}} = \begin{bmatrix} 1-\beta & \beta \end{bmatrix}^\top$,

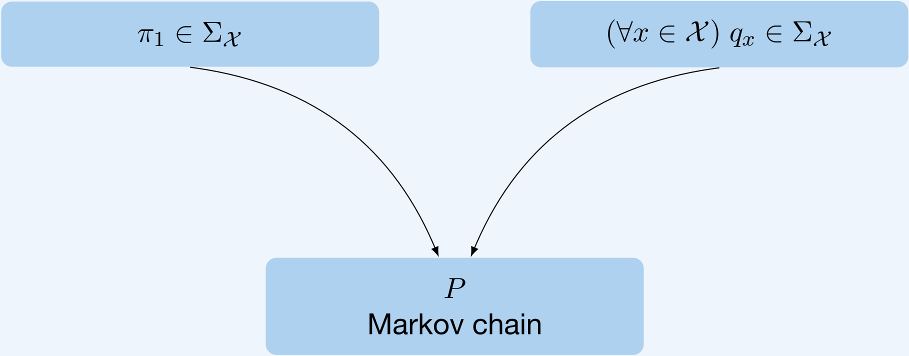
$$\begin{aligned} P(X_3 = \texttt{b}) &= E(\mathbb{I}_{\texttt{b}}(X_3)) \\ &= [\pi_1]_{\mathcal{B}}^\top T^2 \, [\mathbb{I}_b]_{\mathcal{B}} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

Exercise 1

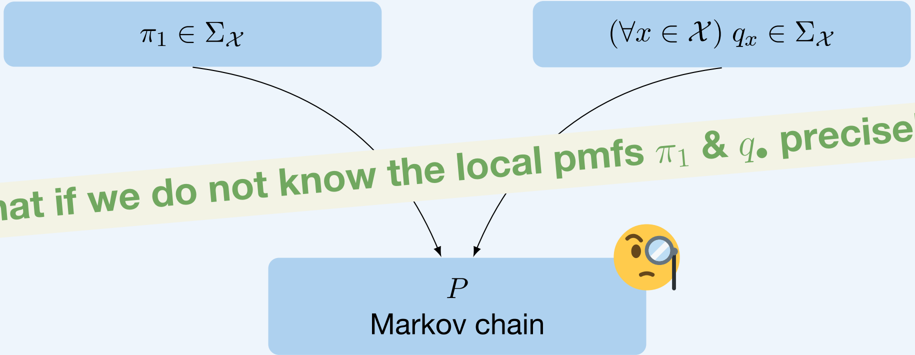
For $\mathcal{X} = \{\mathtt{a}, \mathtt{b}\}$, $\pi_1 = \mathbb{I}_{\mathtt{a}}$, $[q_{\mathtt{a}}]_{\mathcal{B}} = \begin{bmatrix}\alpha & 1-\alpha\end{bmatrix}^\top$ and $[q_{\mathtt{b}}]_{\mathcal{B}} = \begin{bmatrix}1-\beta & \beta\end{bmatrix}^\top$,

$$\begin{aligned} P(X_3 = \mathtt{b}) &= E(\mathbb{I}_{\mathtt{b}}(X_3)) \\ &= [\pi_1]_{\mathcal{B}}^\top T^2 \, [\mathbb{I}_b]_{\mathcal{B}} \\ &= \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}\alpha & 1-\alpha \\ 1-\beta & \beta\end{bmatrix}^2 \begin{bmatrix}0 \\ 1\end{bmatrix} \\ &= \alpha(1-\alpha) + (1-\alpha)\beta. \end{aligned}$$

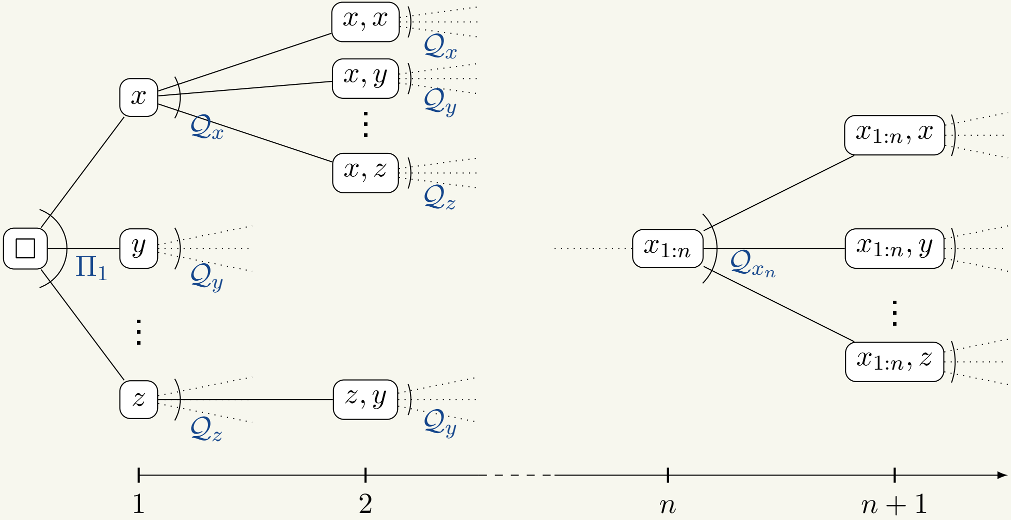
Markov chain



Markov chain



What if we do not know the local pmfs π_1 & $q_\bullet$ precisely?



$\square$
Π_1
x
y
z
$\mathcal{Q}_x$
$\mathcal{Q}_y$
$\mathcal{Q}_z$
x, x
x, y
x, z
z, y
$\mathcal{Q}_x$
$\mathcal{Q}_y$
$\mathcal{Q}_z$
$\mathcal{Q}_y$
$x_{1:n}$
$\mathcal{Q}_{x_n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
1
2
n
$n+1$

Imprecise Markov chain

$\Pi_1 \subseteq \Sigma_{\mathcal{X}}$

$(\forall x \in \mathcal{X})\ \mathcal{Q}_x \subseteq \Sigma_{\mathcal{X}}$

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$\mathcal{P}^{\mathsf{M}}$
all compatible
Markov chains

Upper expectations

For $n \in \mathbb{N}$ and $f \colon \mathcal{X}^n \to \mathbb{R}$,

$$\overline{E}^{\mathsf{M}}\big(f(X_{1:n})\big) := \sup_{P \in \mathcal{P}^{\mathsf{M}}} E_P\big(f(X_{1:n})\big).$$

Upper expectations

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In particular, for $n > 1$ and $f \colon \mathcal{X} \to \mathbb{R}$,

$$\begin{aligned}\overline{E}^{\mathsf{M}}\big(f(X_n)\big) &= \sup_{P \in \mathcal{P}^{\mathsf{M}}} E_P\big(f(X_n)\big)\\ &= \sup_{P \in \mathcal{P}^{\mathsf{M}}} E_{\pi_1}\left(\mathsf{T}^{n-1} f\right).\end{aligned}$$

Exercise 2

Again consider $\mathcal{X} = \{\mathtt{a}, \mathtt{b}\}$, but now with $\Pi_1 = \{\mathbb{I}_{\mathtt{a}}\}$,

$$\mathcal{Q}_{\mathtt{a}} = \{\alpha\mathbb{I}_{\mathtt{a}} + (1-\alpha)\mathbb{I}_{\mathtt{b}} \colon \alpha \in [\underline{\alpha}, \overline{\alpha}]\} \quad \text{and} \quad \mathcal{Q}_{\mathtt{b}} = \{(1-\beta)\mathbb{I}_{\mathtt{a}} + \beta\mathbb{I}_{\mathtt{b}} \colon \beta \in [\underline{\beta}, \overline{\beta}]\},$$

where $\underline{\alpha} = 2/5$, $\overline{\alpha} = 3/5$, $\underline{\beta} = 1/3$ and $\overline{\beta} = 1/2$. Determine

$$\overline{E}^{\mathsf{M}}\big(\mathbb{I}_{\mathtt{b}}(X_3)\big).$$

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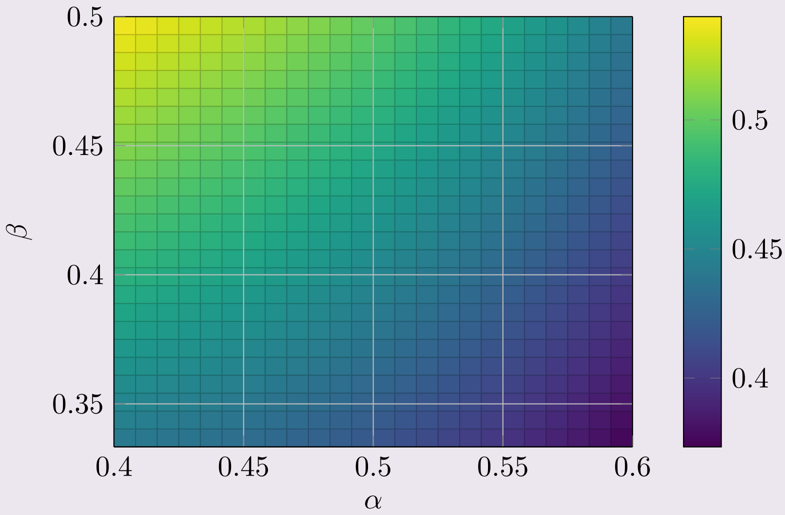
0.5
0.45
0.4
0.35
β
0.4
0.45
0.5
0.55
0.6
α
0.5
0.45
0.4

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$\mathcal{P}^{\mathrm{M}}$
all compatible Markov chains

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$\mathcal{P}^{\mathrm{M}}$
all compatible Markov chains

$\subseteq$

$\mathcal{P}$
all compatible probability trees

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$\mathcal{P}^{\mathrm{M}}$ all compatible Markov chains	$\subseteq$	$\mathcal{P}$ all compatible probability trees
$\overline{E}^{\mathrm{M}}$	$\leq$	$\overline{E}$

Exercise 3

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with $\underline{\alpha} = 2/5$, $\overline{\alpha} = 3/5$, $\underline{\beta} = 1/3$ and $\overline{\beta} = 1/2$. Determine

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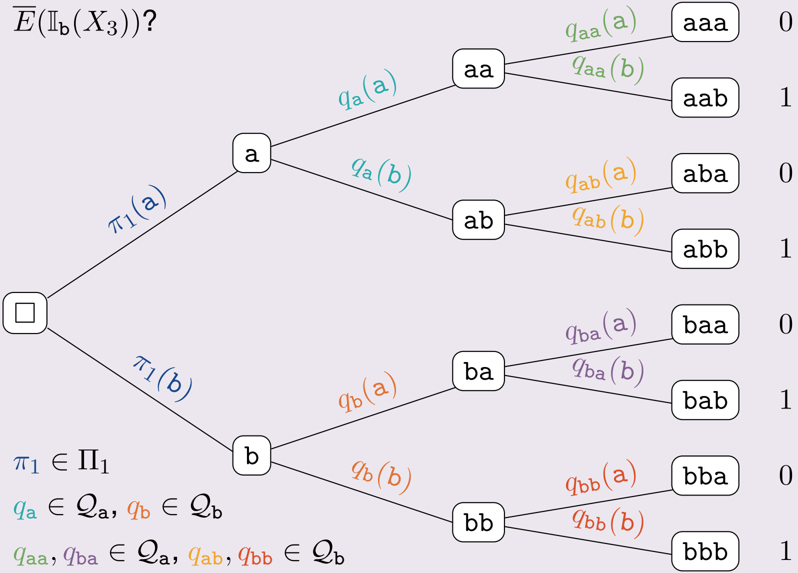
$$\overline{E}\big(\mathbb{I}_{\mathtt{b}}(X_3)\big) = \sup_{P \in \mathcal{P}} E_P\big(\mathbb{I}_{\mathtt{b}}(X_3)\big).$$

$\overline{E}(\mathbb{I}_{\mathtt{b}}(X_3))$?

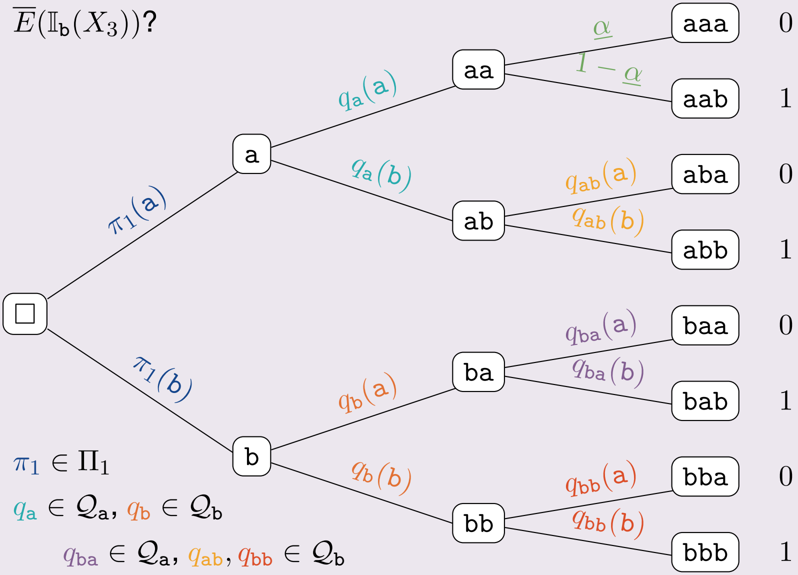


$\pi_1 \in \Pi_1$

$q_{\mathtt{a}} \in \mathcal{Q}_{\mathtt{a}}$, $q_{\mathtt{b}} \in \mathcal{Q}_{\mathtt{b}}$

$q_{\mathtt{aa}}, q_{\mathtt{ba}} \in \mathcal{Q}_{\mathtt{a}}$, $q_{\mathtt{ab}}, q_{\mathtt{bb}} \in \mathcal{Q}_{\mathtt{b}}$

$\overline{E}(\mathbb{I}_{\mathtt{b}}(X_3))$?



$\pi_1 \in \Pi_1$

$q_{\mathtt{a}} \in \mathcal{Q}_{\mathtt{a}}$, $q_{\mathtt{b}} \in \mathcal{Q}_{\mathtt{b}}$

$q_{\mathtt{ba}} \in \mathcal{Q}_{\mathtt{a}}$, $q_{\mathtt{ab}}, q_{\mathtt{bb}} \in \mathcal{Q}_{\mathtt{b}}$

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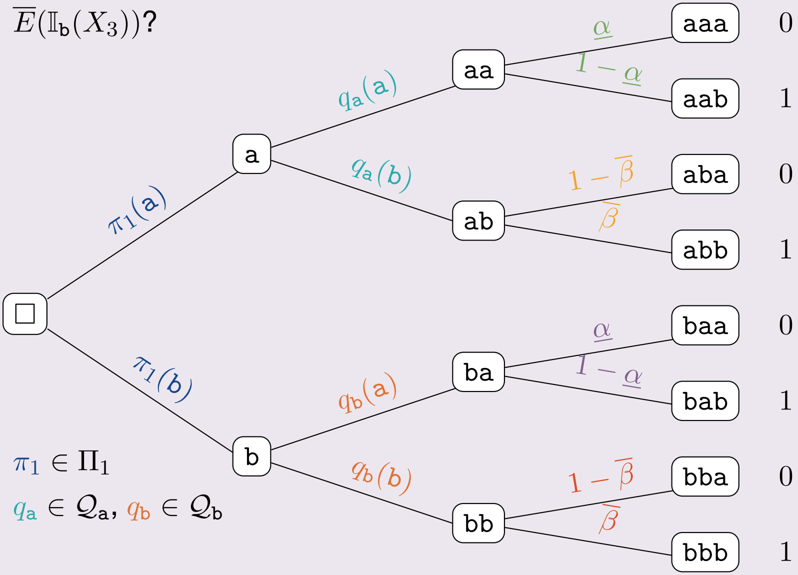

$\pi_1 \in \Pi_1$

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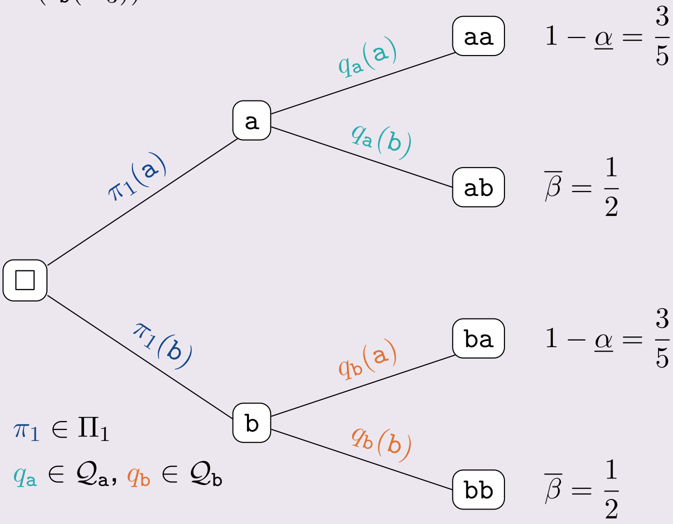

$\pi_1 \in \Pi_1$

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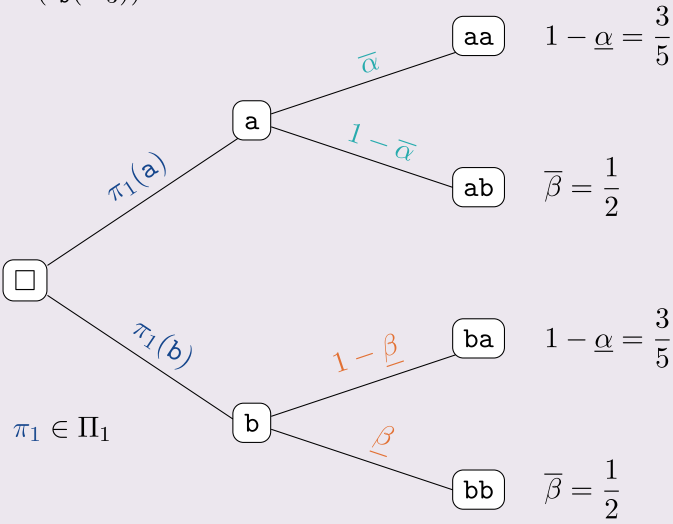
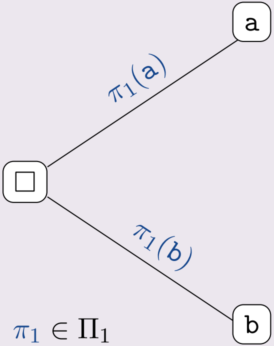


$\pi_1 \in \Pi_1$

$\overline{E}(\mathbb{I}_{\texttt{b}}(X_3))$?



$\overline{\alpha}(1-\underline{\alpha})+(1-\overline{\alpha})\overline{\beta}$

$(1-\underline{\beta})(1-\underline{\alpha})+\underline{\beta}\overline{\beta}$

$\pi_1 \in \Pi_1$

$\overline{E}(\mathbb{I}_{\mathsf{b}}(X_3))$?

☐ $$\overline{E}\big(\mathbb{I}_{\mathsf{b}}(X_3)\big) = \overline{\alpha}(1-\underline{\alpha}) + (1-\overline{\alpha})\overline{\beta} = \frac{28}{50}$$

Exercise 3

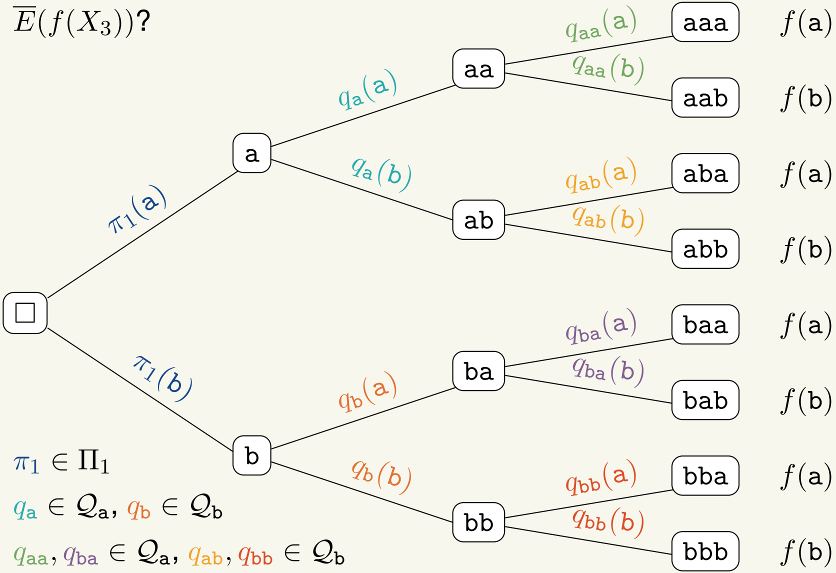
As before consider $\mathcal{X} = \{\mathtt{a}, \mathtt{b}\}$, $\Pi_1 = \{\mathbb{I}_{\mathtt{a}}\}$,

$$\mathcal{Q}_{\mathtt{a}} = \{\alpha\mathbb{I}_{\mathtt{a}} + (1-\alpha)\mathbb{I}_{\mathtt{b}} \colon \alpha \in [\underline{\alpha}, \overline{\alpha}]\} \quad \text{and} \quad \mathcal{Q}_{\mathtt{b}} = \{(1-\beta)\mathbb{I}_{\mathtt{a}} + \beta\mathbb{I}_{\mathtt{b}} \colon \beta \in [\underline{\beta}, \overline{\beta}]\}$$

with $\underline{\alpha} = 2/5$, $\overline{\alpha} = 3/5$, $\underline{\beta} = 1/3$ and $\overline{\beta} = 1/2$. Then

$$\overline{E}\big(\mathbb{I}_{\mathtt{b}}(X_3)\big) = \frac{28}{50} > \frac{27}{50} = \overline{E}^{\mathsf{M}}\big(\mathbb{I}_{\mathtt{b}}(X_3)\big)!$$

$\overline{E}(f(X_3))$?



$\pi_1 \in \Pi_1$

$q_a \in \mathcal{Q}_a$, $q_b \in \mathcal{Q}_b$

$q_{aa}, q_{ba} \in \mathcal{Q}_a$, $q_{ab}, q_{bb} \in \mathcal{Q}_b$

The upper transition operator

Let $\overline{\mathsf{T}} \colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ be the operator which maps $f \colon \mathcal{X} \to \mathbb{R}$ to

$$\overline{\mathsf{T}} f \colon \mathcal{X} \to \mathbb{R} \colon x \mapsto \overline{E}_{\mathcal{Q}_x}(f) = \sup_{q \in \mathcal{Q}_x} \sum_{y \in \mathcal{X}} q(y) f(y).$$

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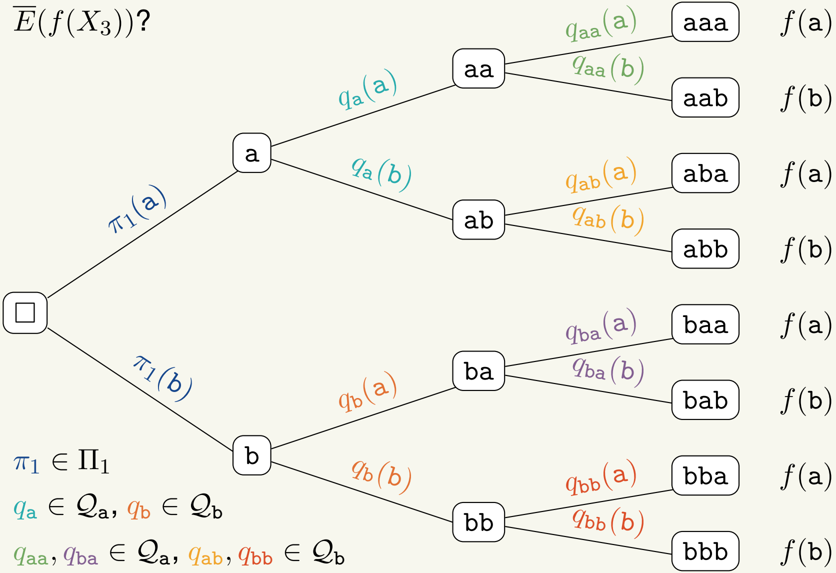
This is an example of an *upper transition operator*: an operator $\overline{S} \colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ such that

1. $\overline{S} f \leq \sup f$ for all $f \in \mathbb{R}^{\mathcal{X}}$;
2. $\overline{S}(f+g) \leq \overline{S} f + \overline{S} g$ for all $f, g \in \mathbb{R}^{\mathcal{X}}$;
3. $\overline{S}(\lambda f) = \lambda \overline{S} f$ for all $\lambda \in \mathbb{R}_{\geq 0}, f \in \mathbb{R}^{\mathcal{X}}$.

$\overline{E}(f(X_3))$?



$\pi_1 \in \Pi_1$

$q_a \in \mathcal{Q}_a$, $q_b \in \mathcal{Q}_b$

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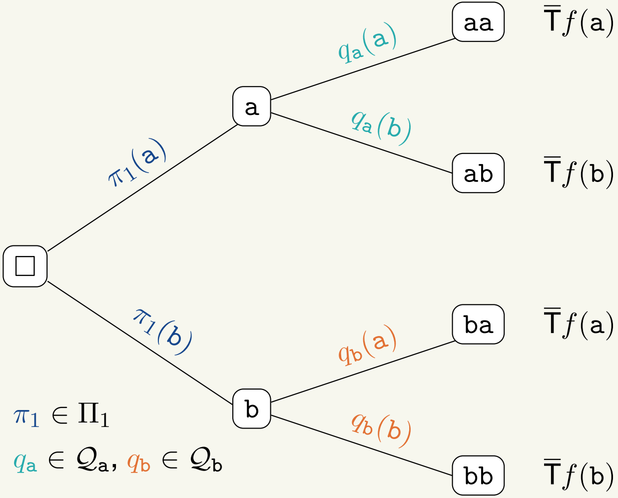
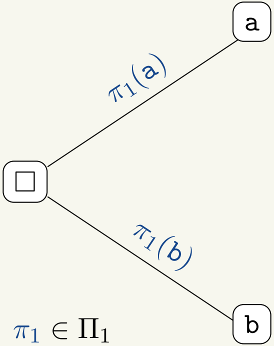
$\overline{E}(f(X_3))$?



$\pi_1 \in \Pi_1$

$q_a \in \mathcal{Q}_a$, $q_b \in \mathcal{Q}_b$

$\overline{E}(f(X_3))$?



$$\left[\overline{\mathsf{T}}\left(\overline{\mathsf{T}}f\right)\right](\mathtt{a}) = \overline{\mathsf{T}}^2 f(\mathtt{a})$$

$$\left[\overline{\mathsf{T}}\left(\overline{\mathsf{T}}f\right)\right](\mathtt{b}) = \overline{\mathsf{T}}^2 f(\mathtt{b})$$

$\pi_1 \in \Pi_1$

$\overline{E}(f(X_3))$?

□ $$\overline{E}\big(f(X_3)\big) = \overline{E}_{\Pi_1}\big(\overline{\mathsf{T}}^2 f\big)$$

Upper expectations

In essence, we've shown that for all $n > 1$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$\overline{E}\big(f(X_n)\big) = \overline{E}_{\Pi_1}\big(\overline{\mathsf{T}}^{n-1} f\big).$$

Upper expectations

In essence, we've shown that for all $n > 1$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$\overline{E}\big(f(X_n)\big) = \overline{E}_{\Pi_1}\big(\overline{\mathsf{T}}^{n-1} f\big).$$

More generally, it has been shown that for all $n > 1$ and $f\colon \mathcal{X}^n \to \mathbb{R}$,

$$\overline{E}\big(f(X_{1:n})\big) = \overline{E}\big([\overline{\mathsf{T}} f(X_{1:n-1}, \bullet)](X_{n-1})\big).$$

Imprecise Markov chains

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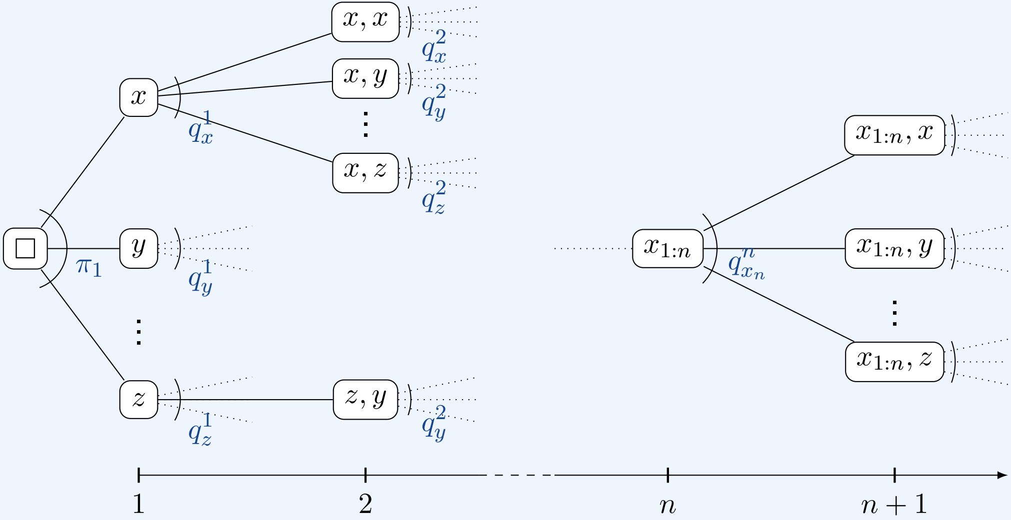
$\mathcal{P}^{\mathrm{M}}$
all compatible
Markov chains

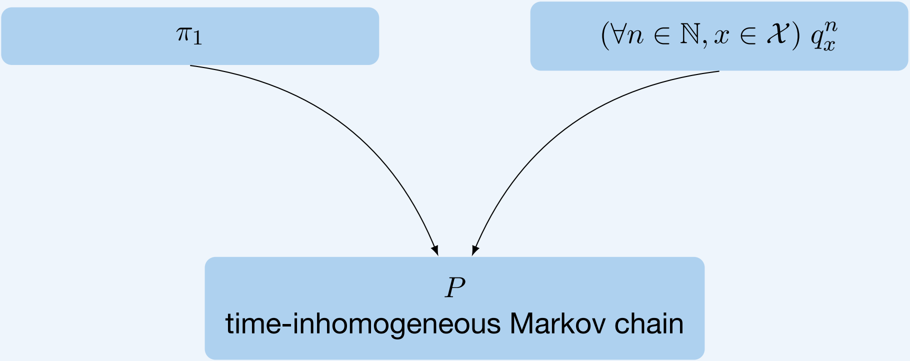
$\subseteq$

$\mathcal{P}$
all compatible
probability trees



x
y
z
x, x
x, y
x, z
z, y
π_1
q_x^1
q_y^1
q_z^1
q_x^2
q_y^2
q_z^2
q_y^2
$x_{1:n}$
$x_{1:n}, x$
$x_{1:n}, y$
$x_{1:n}, z$
$q_{x_n}^n$
1
2
n
n + 1

Time-inhomogeneous Markov chain



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$\mathcal{P}^{\mathrm{M}}$ all compatible Markov chains	$\subseteq$	$\mathcal{P}^{\mathrm{IM}}$ all compatible (time-) inhomogeneous Markov chains	$\subseteq$	$\mathcal{P}$ all compatible probability trees
$\overline{E}^{\mathrm{M}}$	$\leq$	$\overline{E}^{\mathrm{IM}}$	$\leq$	$\overline{E}$

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$\mathcal{P}^{\mathrm{M}}$ all compatible Markov chains	$\subseteq$	$\mathcal{P}^{\mathrm{IM}}$ all compatible (time-) inhomogeneous Markov chains	$\subseteq$	$\mathcal{P}$ all compatible probability trees
$\overline{E}^{\mathrm{M}}$	$\leq$	$\overline{E}^{\mathrm{IM}}$ 🤔	$\leq$	$\overline{E}$

Exercise 4

Verify that for all $f\colon \mathcal{X} \to \mathbb{R}$,

$$\overline{E}^{\mathsf{M}}\big(f(X_1)\big) = \overline{E}^{\mathsf{IM}}\big(f(X_1)\big) = \overline{E}\big(f(X_1)\big).$$

Is it true that for all $n \in \mathbb{N}$ and $f\colon \mathcal{X} \to \mathbb{R}$,

$$\overline{E}^{\mathsf{IM}}\big(f(X_{n+1})\big) = \overline{E}_{\Pi_1}(\overline{\mathsf{T}}^n f) = \overline{E}\big(f(X_{n+1})\big)?$$

More generally, is it true that for all $n > 1$ and $f\colon \mathcal{X}^n \to \mathbb{R}$,

$$\overline{E}^{\mathsf{IM}}\big(f(X_{1:n})\big) = \overline{E}^{\mathsf{IM}}\big([\overline{\mathsf{T}}f(X_{1:n-1,\bullet})](X_{n-1}))\big)?$$

Imprecise Markov chains

$$\Pi_1 \subseteq \Sigma_{\mathcal{X}}$$

$$(\forall x \in \mathcal{X})\ \mathcal{Q}_x \subseteq \Sigma_{\mathcal{X}}$$

Imprecise Markov chains

$$\Pi_1 \subseteq \Sigma_{\mathcal{X}}$$

↓

$$\overline{Q}_1 : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$$

$$(\forall x \in \mathcal{X})\ \mathcal{Q}_x \subseteq \Sigma_{\mathcal{X}}$$

↓

$$(\forall x \in \mathcal{X})\ \overline{Q}_x : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$$

Imprecise Markov chains

$\Pi_1 \subseteq \Sigma_{\mathcal{X}}$

$(\forall x \in \mathcal{X})\ \mathcal{Q}_x \subseteq \Sigma_{\mathcal{X}}$

$\overline{Q}_1 : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$

$(\forall x \in \mathcal{X})\ \overline{Q}_x : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$

Game-theoretic upper expectation

Imprecise Markov chains

$\Pi_1 \subseteq \Sigma_{\mathcal{X}}$

↓

$\overline{Q}_1 : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$

$(\forall x \in \mathcal{X})\ \mathcal{Q}_x \subseteq \Sigma_{\mathcal{X}}$

↓

$\overline{\mathsf{T}} : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$

Imprecise Markov chains

$\Pi_1 \subseteq \Sigma_{\mathcal{X}}$

$\downarrow$

$\overline{Q}_1 \colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$

$(\forall x \in \mathcal{X})\ \mathcal{Q}_x \subseteq \Sigma_{\mathcal{X}}$

$\downarrow$

$\overline{\mathsf{T}} \colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$

Unique upper expectation $\overline{E}$ on $\{f(X_{1:n}) \colon n \in \mathbb{N}, f \in \mathcal{X}^n\}$ such that

(i) $\overline{E}\big(g(X_1)\big) = \overline{Q}_1(g)$ for all $g \in \mathbb{R}^{\mathcal{X}}$

(ii) $\overline{E}\big(h(X_{1:n})\big) = \overline{E}\big([\overline{\mathsf{T}}h(X_{1:n-1}, \bullet)](X_{n-1})\big)$ for all $n > 1$, $h \in \mathcal{X}^n$.

Ergodicity

A transition operator T is *ergodic* if for all $f : \mathcal{X} \to \mathbb{R}$, $\mathrm{T}^n f$ converges to a constant function as $n \to \infty$.

Ergodicity

A transition operator T is *ergodic* if for all $f\colon \mathcal{X} \to \mathbb{R}$, $\mathsf{T}^n f$ converges to a constant function as $n \to \infty$.

In that case, for any $\pi_1 \in \Sigma_{\mathcal{X}}$,

$$E\big(f(X_n)\big) = E_{\pi_1}(\mathsf{T}^{n-1} f)$$

converges to this constant value as $n \to \infty$.

Necessary and sufficient conditions for ergodicity are well-known.

Ergodicity

An upper transition operator $\overline{\mathsf{T}}$ is *ergodic* if for all $f\colon \mathcal{X} \to \mathbb{R}$, $\overline{\mathsf{T}}^n f$ converges to a constant function as $n \to \infty$.

In that case, for any $\Pi_1 \subseteq \Sigma_{\mathcal{X}}$,

$$\overline{E}\big(f(X_n)\big) = \overline{E}_{\Pi_1}(\overline{\mathsf{T}}^{n-1} f)$$

converges to this constant value as $n \to \infty$.

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Necessary and sufficient conditions for ergodicity have been studied,
but the aim of the project is to rediscover them!

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