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The logic of desirability (?)

Some observations on (abstract) logics and the theory of desirability

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SIPTA 2024
Ghent, August 12-16

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Who and why?

Alessio Benavoli (the real mad hatter)

A (convenience*? / former) logician

*Conception and formulation by Yoichi Hirai

Marco Zaffalon: “TDG *is* logic”

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Who and why?

2.3 Connection with Classical Propositional Logic

The definition of a coherent set of desirable gambles, and Theorem 1, make clear that inference with desirable gambles bears a formal resemblance to deduction in classical proposition logic: D3 is a production axiom that states that positive linear combinations of desirable gambles are again desirable. The exact correspondences are listed in the following table:

Classical propositional logic	Sets of desirable gambles
logical consistency	avoiding non-positivity
deductively closed	coherent
deductive closure	natural extension

2.4. Deductive Extension and Deductive Closure

Based on the assumption that the gamble pay-offs are expressed in a linear precise utility scale, statements acceptance imply other statements, generated by positive scaling and combination: if f is judged acceptable, th $\lambda \cdot f$ should be as well for all real $\lambda > 0$; if f and g are judged acceptable, then $f+g$ should be as well. This is call ***deductive extension***. Deductive extension can be succinctly expressed using the positive linear hull operator p which generates convex cones and was introduced in Section 1.4. The assumptions about the utility scale have no dir consequences for reject statements; their *indirect* impact will be derived in Section 2.5.

So, starting from an assessment $\mathcal{A}$ in $\mathbf{A}$, its deductive extension $\mathrm{ext}_{\mathbf{D}}\mathcal{A} := \langle \mathrm{posi}\,\mathcal{A}_{\succeq}; \mathcal{A}_{\prec}\rangle$, which we call a *deductiv closed* assessment, can be derived. Deductively closed assessments $\mathcal{D}$ satisfy the following rationality axiom:

***Axiom* DC** (Deductive Closure). $\quad \mathrm{ext}_{\mathbf{D}}\mathcal{D} = \mathcal{D}$ or, equivalently, $\mathcal{D}_{\succeq} \in \mathrm{C}$,

which can also be expressed as the combination of

***Axiom* PS** (Positive Scaling). $\quad \lambda > 0 \wedge f \in \mathcal{D}_{\succeq} \Rightarrow \lambda \cdot f \in \mathcal{D}_{\succeq}$ or, equivalently, $\mathbb{R}_{>} \cdot \mathcal{D}_{\succeq} \subseteq \mathcal{D}_{\succeq}$

and

***Axiom* C** (Combination). $\quad f, g \in \mathcal{D}_{\succeq} \Rightarrow f + g \in \mathcal{D}_{\succeq}$ or, equivalently, $\mathcal{D}_{\succeq} + \mathcal{D}_{\succeq} \subseteq \mathcal{D}_{\succeq}$.

The subset of $\mathbf{A}$ consisting of all deductively closed assessments is—not surprisingly—denoted by $\mathbf{D}$ and those with confusion by $\mathbb{D} := \mathbf{D} \cap \mathbb{A}$. Not all assessments without confusion remain so after deductive extension; those that do called *deductively closable* and form the set $\mathbb{A}^{+} := \{\mathcal{A} \in \mathbb{A} : \mathrm{ext}_{\mathbf{D}}\mathcal{A} \in \mathbb{D}\}$, where we have made use of the fact that ex never removes statements and therefore cannot remove confusion.

It is useful to have an explicit criterion on hand to test whether an assessment is deductively closable or not:

***Theorem* 2.2.** An assessment $\mathcal{A}$ in $\mathbf{A}$ is deductively closable—i.e., $\mathcal{A} \in \mathbb{A}^{+}$—if and only if $0 \notin \mathcal{A}_{\prec} - \mathrm{posi}\,\mathcal{A}_{\succeq}$.

This criterion is a feasibility problem. When the assessment consists of only a finite number of statements, the feasi space $\mathcal{A}_{\prec} - \mathrm{posi}\,\mathcal{A}_{\succeq}$ is a union of convex cones $\bigcup_{f \in \mathcal{A}_{\prec}} (\{f\} - \mathrm{posi}\,\mathcal{A}_{\succeq})$ and the problem becomes a disjunctive lin program. It reduces to a plain linear program when, for example, $\mathcal{A}_{\prec}$ is also convex.

Again, it is possible to automatically remove confusion from deductively closed assessments, but there is l flexibility than for assessments because not all modified assessments suggested in Proposition 2.1 are deductiv closable:

Proposition 2.3. While ensuring the resulting assessment is still deductively closed, confusion can be removed fro deductively closed assessment $\mathcal{D}$ in $\mathbf{D}$, by removing the confusing gambles from the rejected gambles or by removi ... umbles and then taking the deductive extension. So formally we ha

... $-\mathcal{D}_{\succeq}$ of a deductively closed assessment $\mathcal{D}$ in $\mathbf{D}$ is the negation invari ... pty or a linear space, the cone's so-called *lineality space*. The lineal ... s:

... ssessment $\mathcal{D}$ in $\mathbf{D}$ with a non-empty set of indifferent gambles $\mathcal{D}_{\simeq}$, den ... by $\mathcal{D}_{\succeq,\not\simeq} := \mathcal{D}_{\succeq} \setminus \mathcal{D}_{\simeq}$, then ... ski addition with the indifferent gambles: $\mathcal{D}_{\succeq,\not\simeq} + \mathcal{D}_{\simeq} = \mathcal{D}_{\succeq,\not\simeq}$, and

2. Belief structures

2.1. Classical propositional logic

Consider an object language $\mathbf{L}$ of well-formed formulae, or sentences, in classical propositional logic with the usual axiomatisation (see for instance [6, 28]). We call any subset K of $\mathbf{L}$, *i.e.*, any set of sentences, a *belief model*.[1] Intuitively, a set of sentences K models the beliefs of a subject: it contains those sentences that the subject is certain are true. Of course, this is a very simple type of model, because it concentrates on certainty, or full belief. We shall want to study more general models, that are also able to represent a

Who and why?

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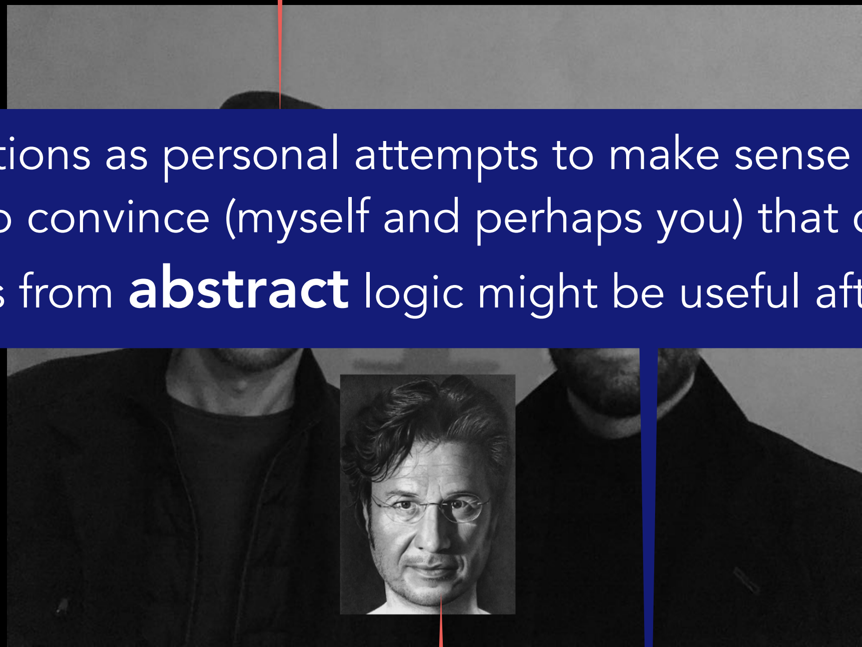

Presentations as personal attempts to make sense of this… and try to convince (myself and perhaps you) that concept and tools from **abstract** logic might be useful after all in IP.

convenience*? /
rmer) logician

eption and formulation by
Hirai

Marco Zaffalon: "TDG *is* logic"

Who and why?

- But also, try to connect with and "make sense" (from a logical point of view) of other existing very nice and interesting stuff within the IP area
  - so the "who" should also include (many) other people, although I will not be able to go very far, and thus acknowledge everyone and their work as I should…

my perspective will be **very** biased, and I will be interested in the abstract perspective from the Polish tradition, hence again, many works and different traditions will be excluded

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Who and why?

A Logical View of Probability

Nic Wilson[1] and **Serafín Moral**[2]

Abstract. Imprecise Probability (or Upper and Lower Probability) is represented as a very simple but powerful logic. Despite having a very different language from classical logics, it enjoys many of the most important properties, which means that some extensions to classical logic can be applied in a fairly straightforward way. The logic is extended to allow qualitative grades of belief, which can be used to represent degrees of caution, and this is applied to create theories of belief revision and non-monotonic inference for probability statements. We also construct a theory of default probability which is based on a variant of Reiter's default logic; this can be used to express and reason with default probability statements.

1 INTRODUCTION

The best understood and most highly developed theory of uncertainty is Bayesian probability. There is a large literature on its foundations and there are many different justifications of the theory; however, all of these assume that for any proposition a, the beliefs in a and $\neg a$ are strongly tied together. Without compelling justification, this assumption greatly restricts the type of information that can be satisfactorily represented, e.g., it makes it impossible to represent adequately partial information about an unknown chance distribution P

Apart from being a simple and elegant way to express this theory of probability, there are other benefits of expressing it as a logic. It brings into the logician's domain this semantically very well founded, fairly well-behaved and expressive representation of beliefs. Because the logic has many of the properties of classical logics, it means that augmentations of classical logic can be applied relatively easily to this logic.

Imprecise Probability does not distinguish caution from ignorance; in section 3 we look at a way of extending the theory to allow qualitative grades, which can be used to represent degrees of caution. Like classical logic, it is very conservative, and is monotonic. It therefore seems natural to look at extensions which tentatively allow stronger conclusions to be drawn, but avoid inconsistency. Three examples of this are given; in section 4, work on belief revision and non-monotonic inference relations is extended to this logic, which leads to ways of resolving inconsistencies, and in section 5, a version of Reiter's Default Logic is applied, which allows more complex tentative assumptions.

2 THE LOGIC OF GAMBLES

Let Ω be a finite set of possibilities, exactly one of which must be true. A gamble on Ω is a function from Ω to $I\!R$. If you were to accept gamble X and ω turned out to be true then you would gain $X(\omega)$ utiles (so you would lose if $X(\omega) < 0$).

A Probabilistic Logic Based on the Acceptability of Gambles ⋆

Peter R. Gillett [a,∗], Richard B. Scherl [b,1], Glenn Shafer [a]

[a] *Rutgers Business School—Newark and New Brunswick*

[b] *Monmouth University, New Jersey*

Abstract

This article presents a probabilistic logic whose sentences can be interpreted as asserting the acceptability of gambles described in terms of an underlying logic. This probabilistic logic has a concrete syntax and a complete inference procedure, and it handles conditional as well as unconditional probabilities. It synthesizes Nilsson's probabilistic logic and Frisch and Haddawy's anytime inference procedure with Wilson and Moral's logic of gambles.

Two distinct semantics can be used for our probabilistic logic: (1) the measure-theoretic semantics used by the prior logics already mentioned and also by the more expressive logic of Fagin, Halpern, and Meggido and (2) a behavioral semantics. Under the measure-theoretic semantics, sentences of our probabilistic logic are interpreted as assertions about a probability distribution over interpretations of the underlying logic. Under the behavioral semantics, these sentences are interpreted only as asserting the acceptability of gambles, and this suggests different directions for generalization.

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International Journal of Approximate Reasoning
Volume 57, February 2015, Pages 69-102

Accept & reject statement-based uncertainty models

Erik Quaeghebeur [a b 1], Gert de Cooman [a], Filip Hermans [a]

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Highlights

- We develop a framework for modelling and reasoning based on a pair of gamble sets.

Proceedings of Machine Learning Research 147:191–200, 2021 ISIPTA 2021

Information Algebras of Coherent Sets of Gambles in General Possibility Spaces

Juerg Kohlas JUERG.KOHLAS@UNIFR.CH
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Abstract

In this paper, we show that coherent sets of gambles can be embedded into the algebraic structure of *information algebra*. This leads firstly, to a new perspective of the algebraic and logical structure of desirability

the particular case where information one is interested in concerns the values of certain groups of variables $\{X_i : i \in I\}$ with I an index set, $\Omega = \times_{i\in I}\Omega_i$, where Ω_i is the set of possible values of X_i, and $\omega \equiv_S \omega' \iff \omega|_S = \omega'|_S$, [1] for every $S \subseteq I$ and $\omega, \omega' \in \Omega$. (see [12]).

Proceedings of Machine Learning Research 147:42–50, 2021 ISIPTA 2021

Probability Filters as a Model of Belief; Comparisons to the Framework of Desirable Gambles.

Catrin Campbell-Moore CATRIN.CAMPBELL-MOORE@BRISTOL.AC.UK
Department of Philosophy, University of Bristol, UK

Abstract

We propose a model of uncertain belief. This models coherent beliefs by a filter, F, on the set of probabilities. That is, it is given by a collection of sets of probabilities which are closed under supersets and finite intersections. This can naturally capture your probabilistic judgements. When you think that it is more likely to be sunny than rainy, we have $\{p \mid p(\textsc{Sunny}) > p(\textsc{Rainy})\} \in F$. When you think that a gamble g is desirable, we have $\{p \mid \mathrm{Exp}_p[g] > 0\} \in F$.

of the model of choice functions, or sets of desirable gamble sets (we include a mixing axiom, but no Archimedean axiom).

Using other terms, this model was proposed and discussed in a joint paper with Jason Konek, (Campbell-Moore and Konek, 2019), using the interpretation of beliefs with probabilistic contents as outlined in Moss (2018). The main results of this paper were stated there without proof.

The paper proceeds as follows. We introduce the model

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International Journal of Approximate Reasoning
Volume 172, September 2024, 109241

The logic behind desirable sets of things, and its filter representation

Gert de Cooman [a], Arthur Van Camp [b], Jasper De Bock [a]

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Highlights

- We identify the (filter representation of the) logic behind the recent theory of coherent sets of desirable (sets of) things.

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Where?

Whenever possible I will explicitly mention where concepts, tools, and results can be found

What?

- The theory of desirability as a logic;
  - Tools from logic can justify some views on the theory of desirability
  - Probabilistic semantics and completeness
- Extensions of the theory of desirability may be understood and enriched by tools from logic
  - Adding Varieties of rejection
  - The case of accept & reject and the case of intuitionistic rejection

What?

- The theory of d
  - Tools from lo
  - Probabilistic
- Extensions of the theory of desirability may be understood and enriched by tools from logic
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Proposal for project related to study natural logical extension of this setting, and checking possible link with other existing approaches within IP.

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The main underlying question now being:
is the theory of choice function a logic?

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Part I: Logic and desirability

(Abstract) consequences and desirability

What is logic: a lazy start

- The study of (correct) informational processes of inference (reasoning)
  - When, given something that has been asserted / assumed as true, can we assert / assume as true something else?
  - Examples:
    - If the enemy cuts the Sambuco's dam, the Val Lavizzara will be inundated. The Val Lavizzara is not inundated. Hence, the enemy did not cut the Sambuco's dam.
    - If the enemy cuts the Sambuco's dam, the Val Lavizzara will be inundated. The Val Lavizzara is inundated. Hence, the enemy has cut the Sambuco's dam.
    - If Carlo won the race, then, if Mario came second then Sergio came third; Mario did not come second. Hence, either Carlo won or Sergio came third.

What is logic: a lazy start

relation between "things", and its properties

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valid / sound in virtue of the form of considered "things" in the relation

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What is logic: a lazy start

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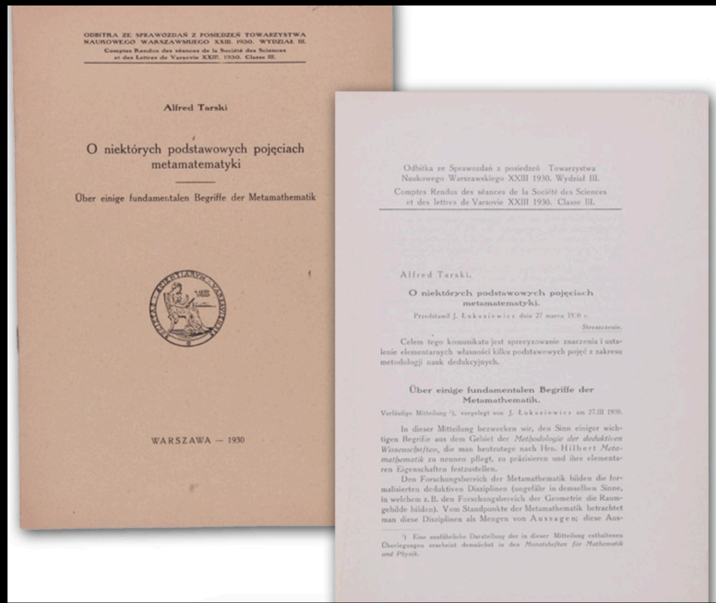
  - Examples:
    - If trimballirin quaqqua, then machin_truc supercazzola. Not machine_truc supercazzola. Hence, trimballirin quaqqua.
    - If trimballirin quaqqua, then machin_truc supercazzola. Machin_truc supercazzola. Hence, trimballirin quaqqua.
    - If Carlo blabla, then, if Mario squaraush then Sergio proprot; not the case that Mario squaraush. Hence, either Carlo blabla or Sergio proprot.

not valid / not sound **in virtue of the form of considered "things" in the relation**

The idea of (abstract) consequence relation

- The study of a general theory of logical systems traces back to the work of Alfred Tarski, Paul Hertz and of Gerhard Gentzen in the early twentieth century.

Gerhard Gentzen (24 November 1909 – 4 August 1945) was a German mathematician and logician. He made major contributions to the foundations of mathematics, proof theory, especially on natural deduction and sequent calculus.



ODBITKA ZE SPRAWOZDAŃ Z POSIEDZEŃ TOWARZYSTWA NAUKOWEGO WARSZAWSKIEGO XXIII. 1930. WYDZIAŁ III.
Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie XXIII. 1930. Classe III.

Alfred Tarski

O niektórych podstawowych pojęciach metamatematyki

Über einige fundamentalen Begriffe der Metamathematik

WARSZAWA — 1930

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Alfred Tarski.

O niektórych podstawowych pojęciach metamatematyki.

Przedstawił J. Łukasiewicz dnia 27 marca 1930 r.

Streszczenie.

Celem tego komunikatu jest sprecyzowanie znaczenia i ustalenie elementarnych własności kilku podstawowych pojęć z zakresu metodologji nauk dedukcyjnych.

Über einige fundamentalen Begriffe der Metamathematik.

Vorläufige Mitteilung [1], vorgelegt von J. Łukasiewicz am 27.III 1930.

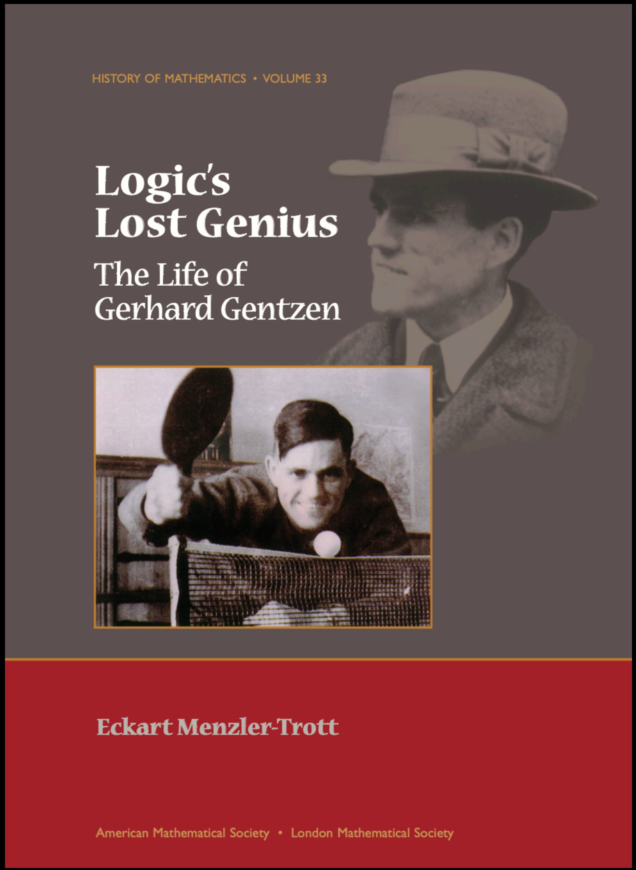
In dieser Mitteilung bezwecken wir, den Sinn einiger wichtigen Begriffe aus dem Gebiet der *Methodologie der deduktiven Wissenschaften*, die man heutzutage nach Hrn. Hilbert *Metamathematik* zu nennen pflegt, zu präzisieren und ihre elementaren Eigenschaften festzustellen.

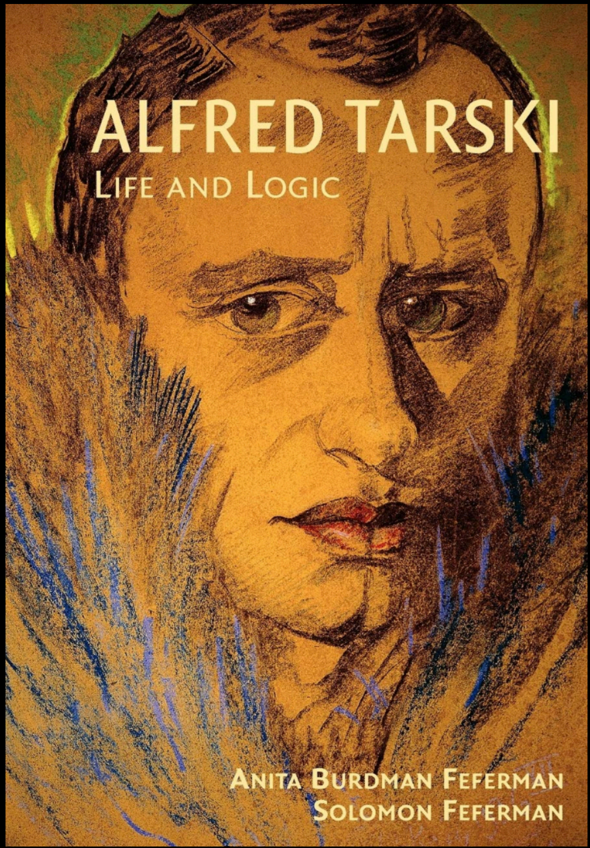
Den Forschungsbereich der Metamathematik bilden die formalisierten deduktiven Disziplinen (ungefähr in demselben Sinne, in welchem z. B. den Forschungsbereich der Geometrie die Raumgebilde bilden). Vom Standpunkte der Metamathematik betrachtet man diese Disziplinen als Mengen von Aussagen; diese Aus-

[1] Eine ausführliche Darstellung der in dieser Mitteilung enthaltenen Überlegungen erscheint demnächst in den *Monatshefte für Mathematik und Physik*.

Alfred Tarski (born Alfred Teitelbaum; January 14, 1901 – October 26, 1983) was a Polish-American logician and mathematician.

A small digression





Varieties of presenting inference

- Unary assertional system: $\vdash \varphi$
- Unary rejection system: $\varphi \vdash$
- Binary implicational system, or "thing-thing" consequence relation : $\gamma \vdash \varphi$
- Asymmetric, or "set (of things) - thing", consequence relation : $\Gamma \vdash \varphi$
- Symmetric, or "set (of things) - set (of things) ", consequence relation : $\Gamma \vdash \Phi$

First, the binary case: *thing-thing*

- Suppose we are given a set A of “things”, assertions, claims, etc.
- We may want to model a relation between “things” such that whenever I assert/accept/consider as true some “thing” a, we should also *necessarily* assert/accept/consider as true some other “thing” b.
  - Stated otherwise, the acceptance of a entails, implies the acceptance of b
- How to characterise such relation?

First, the binary case: *thing-thing*

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  - Stated otherwise, the acceptance of a entails, implies the acceptance of b
- How to characterise such relation? Two ways:
  - *Syntactic* characterisation: provide a list of minimal structural properties / principles / axioms that such a relation should satisfy
  - *Semantic* characterisation: make reference to "something else", external, more "primitive", given by what such "things" are supposed to represent

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"Thing-thing" consequence relation: syntactic characterisation

- What are the minimal property of the relation "the acceptance of a entails the acceptance of b"?

"Thing-thing" consequence relation: syntactic characterisation

- What are the minimal property of the relation "the acceptance of a entails the acceptance of b"?
  - $a \vdash a$ (reflexivity)
  - If $a \vdash b$ and $b \vdash c$, then $a \vdash c$ (transitivity)
- That is:

A "thing-thing" (binary) consequence relation $\vdash \subseteq A \times A$ is a **pre-order** (or quasi-order) over A, and the relational structure $(A, \vdash)$ is a **pre-ordered set** (or quasi-ordered set).

"Thing-thing" consequence relation: syntactic characterisation

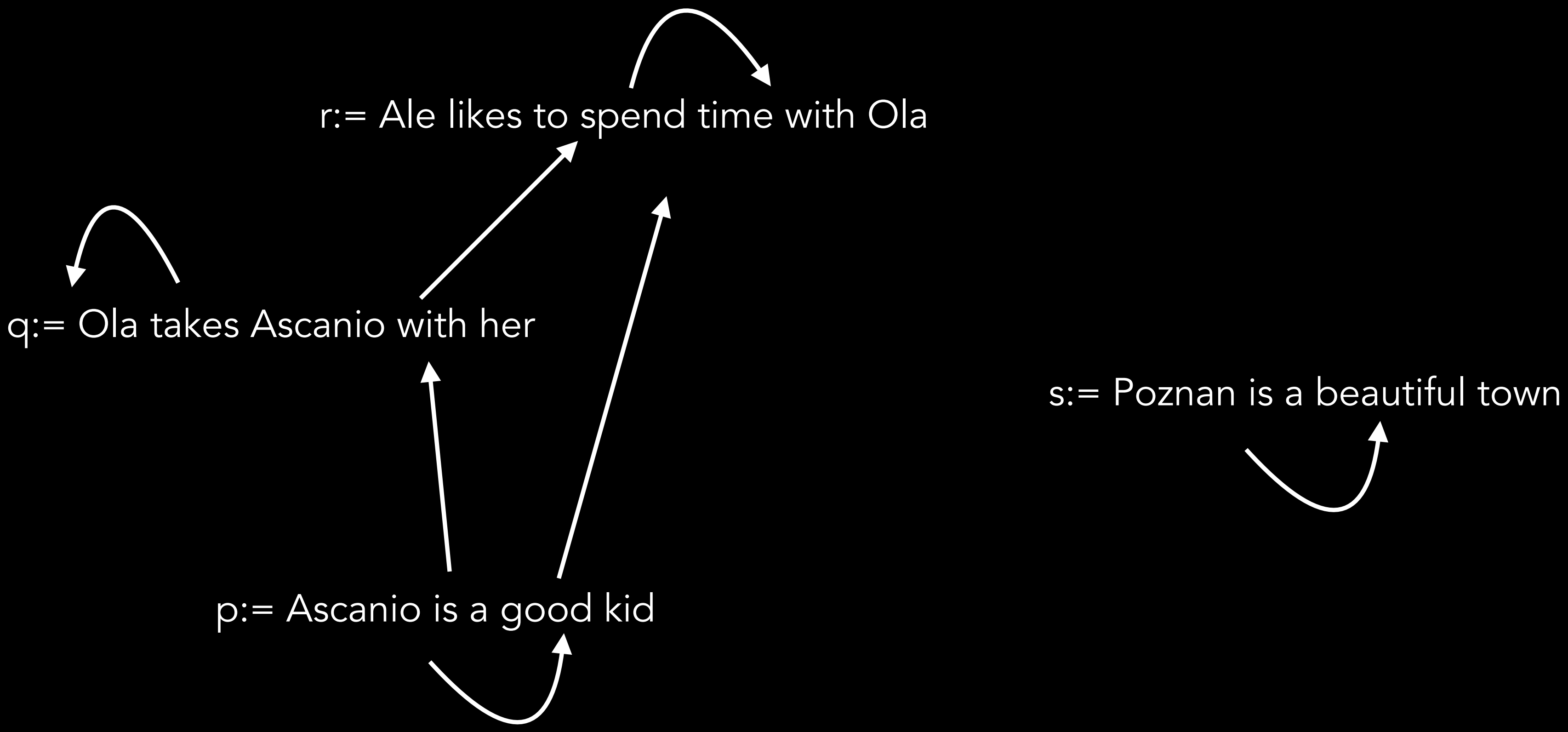
r:= Ale likes to spend time with Ola

q:= Ola takes Ascanio with her

s:= Poznan is a beautiful town

p:= Ascanio is a good kid

"Thing-thing" consequence relation: syntactic characterisation



"Thing-thing" consequence relation: semantic characterisation

Assume a possible world (state of affairs / beliefs) is characterised by the things I accept, i.e. by a (valuation) function $\mathfrak{s} : A \to \{\mathsf{a}, \mathsf{r}\}$.

Thus a possible world is tantamount to the "truth set" $\{a \in A \mid \mathfrak{s}(a) = \mathsf{a}\}$ of its characteristic (valuation) function π, and we will move freely from seeing π as a subset of A or a characteristic function.

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Thus a possible world is tantamount to the "truth set" $\{a \in A \mid \mathfrak{s}(a) = \mathsf{a}\}$ of its characteristic (valuation) function π, and we will move freely from seeing π as a subset of A or a characteristic function.

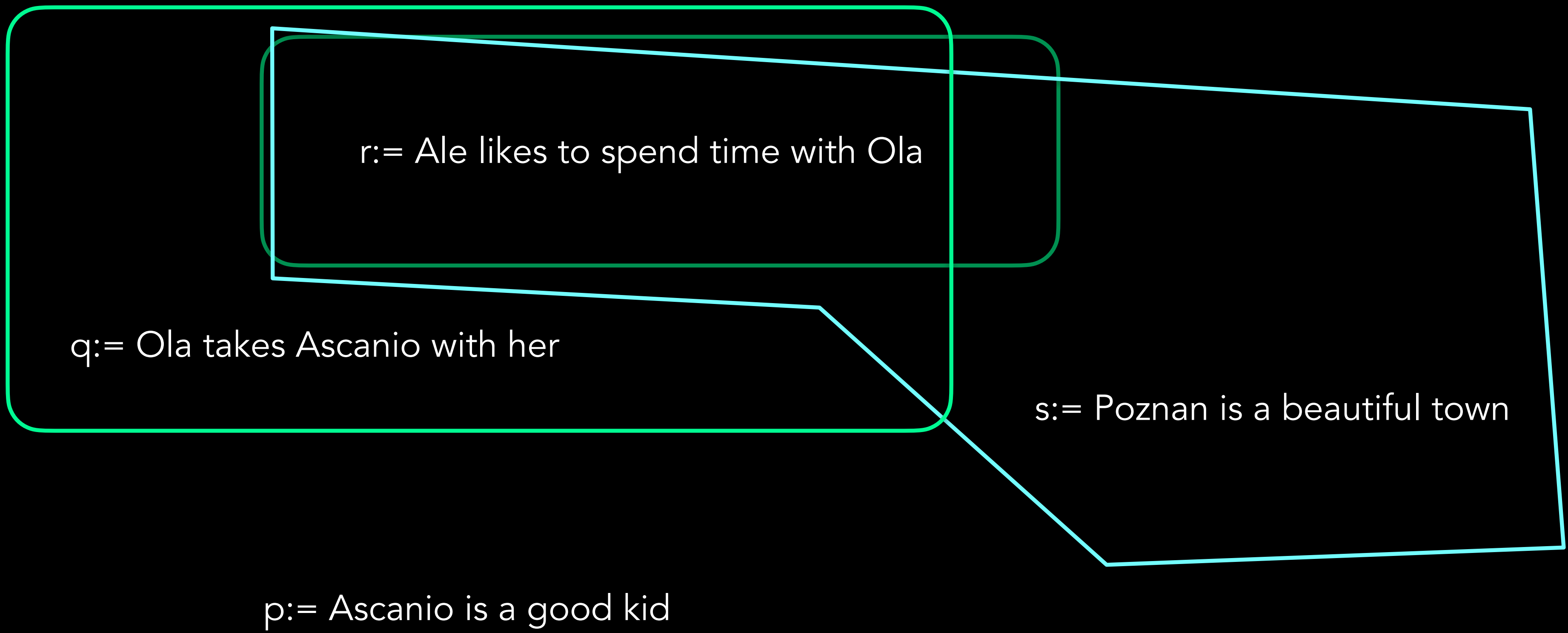
Given a set of possible worlds $\mathfrak{S}$, the collection of possible world in which a "thing" $a \in A$ is true/ accepted / … is defined as $\mathfrak{S}(a) := \{\mathfrak{s} \in \mathfrak{S} \mid a \in \mathfrak{s}\}$

Hence

- *Definition*: The (semantic) relation $\vdash_{\mathfrak{S}}$ generated by $\mathfrak{S}$ over A is defined as

$$a \vdash_{\mathfrak{S}} b \text{ if and only if } \mathfrak{S}(a) \subseteq \mathfrak{S}(b)$$

"Thing-thing" consequence relation: semantic characterisation



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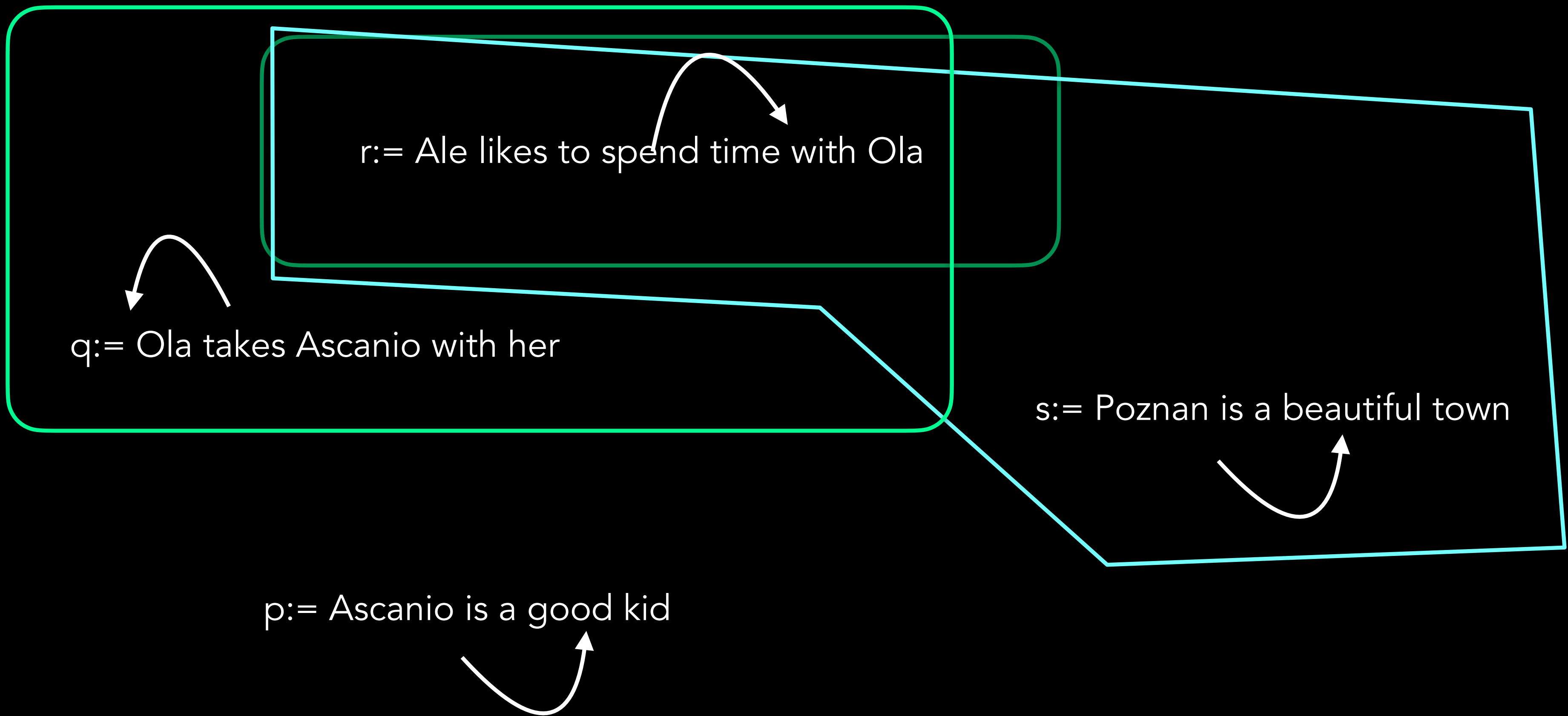

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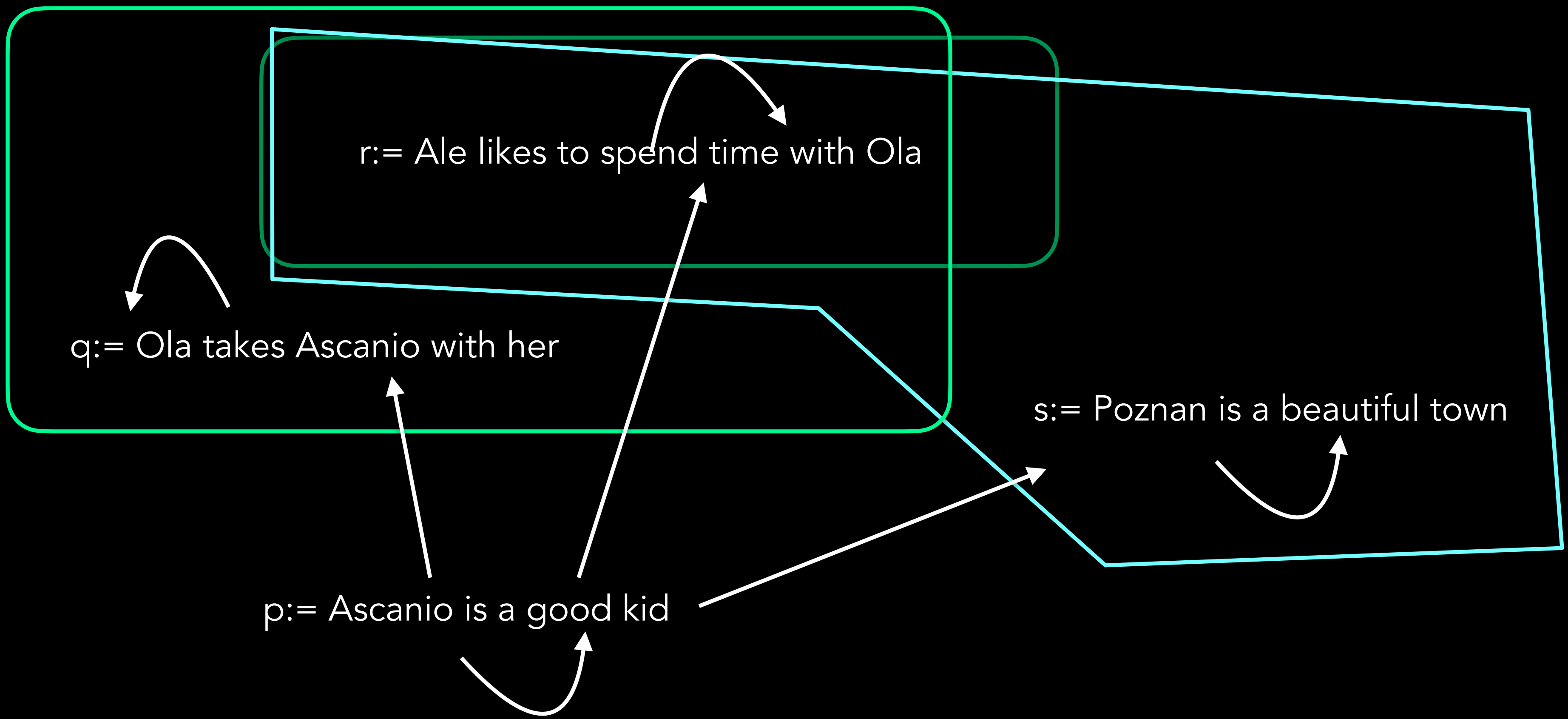

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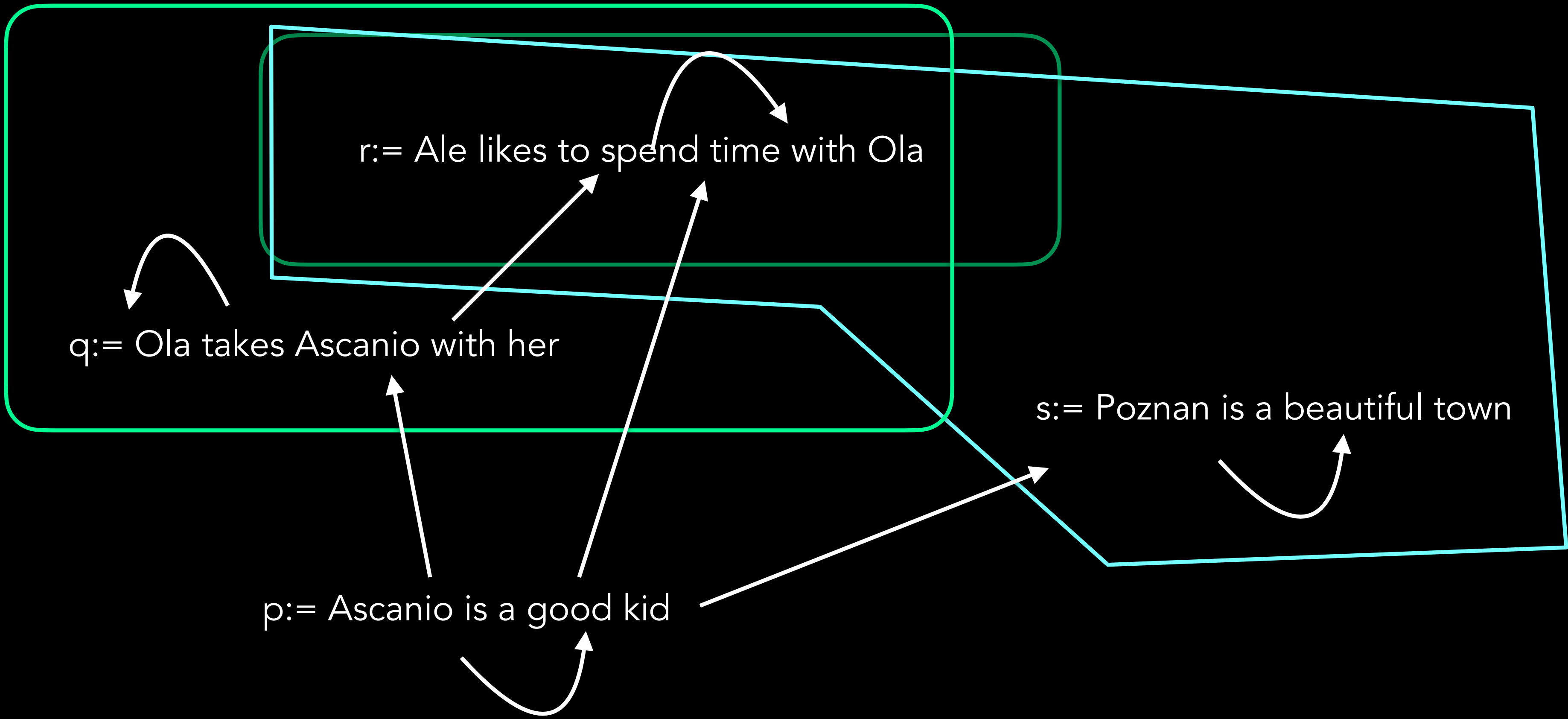
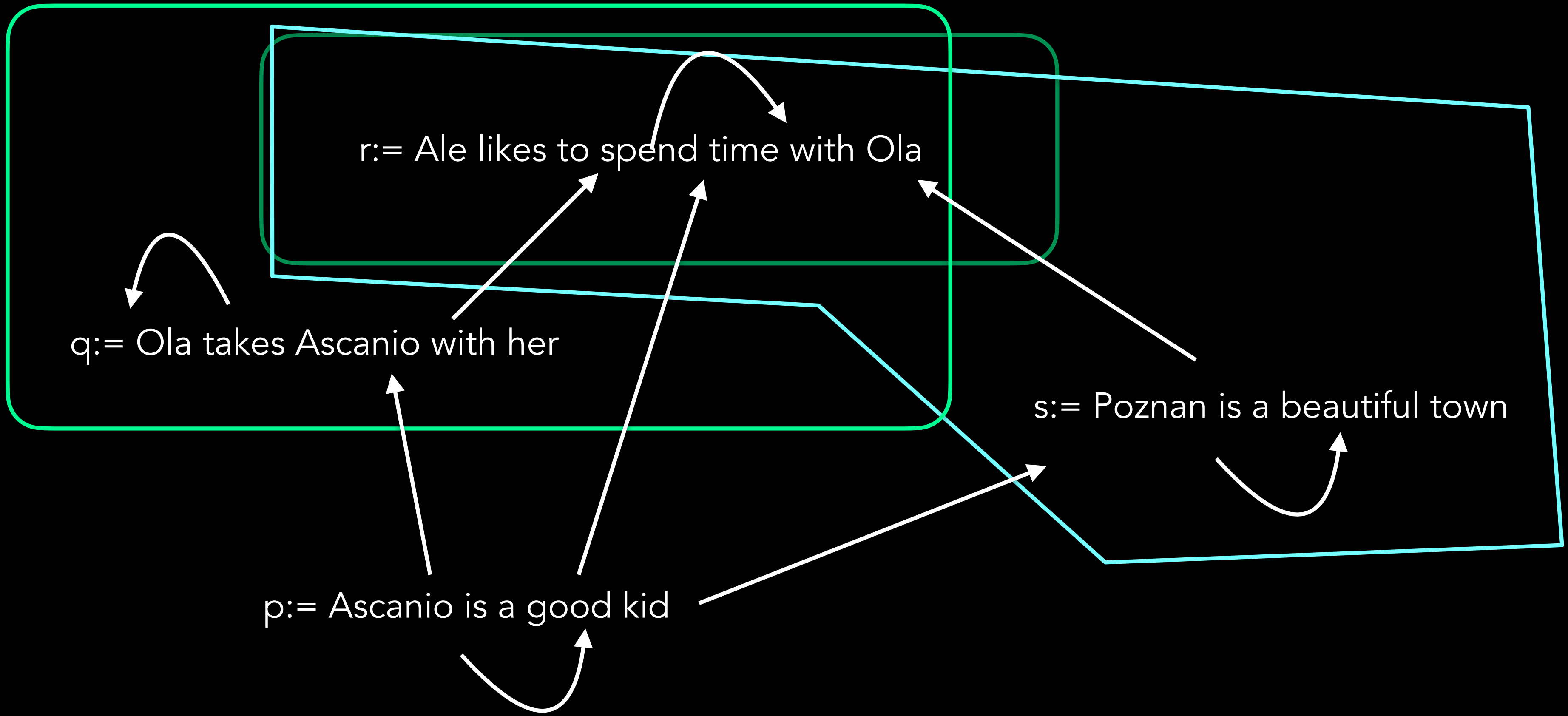


"Thing-thing" consequence relation: semantic characterisation



"Thing-thing" consequence relation: semantic characterisation

It is immediate to verify that

- *Fact*: Given a set of possible worlds $\mathfrak{S}$, the structure $(A, \vdash_{\mathfrak{S}})$ is a pre-ordered set, meaning that $\vdash_{\mathfrak{S}}$ is a binary consequence relation over A.

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Completeness question: Given a binary implication (pre-order) $\vdash$ on some set of "things", is there a class of possible words / valuations $\mathfrak{S}$ over A whose induces semantic relation (pre-order) $\vdash_{\mathfrak{S}}$ coincide with $\vdash$? Is there some kind of "canonical" (and "concrete") semantics $\mathfrak{S}$ representing (inducing) $\vdash$?

"Thing-thing" consequence relation: an abstract completeness theorem

- *Question*: Given a binary implication $\vdash$, is there a class of possible words $\mathfrak{S}$ over A whose semantic relation $\vdash_{\mathfrak{S}}$ coincide with the relation $\vdash$?

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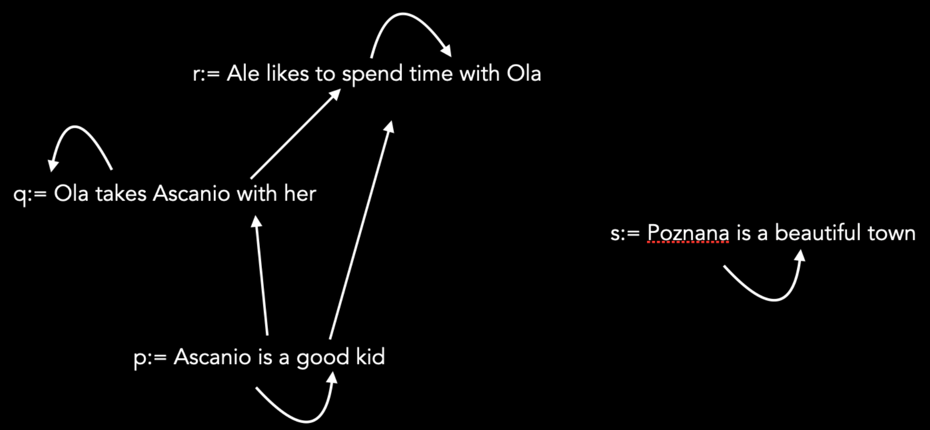

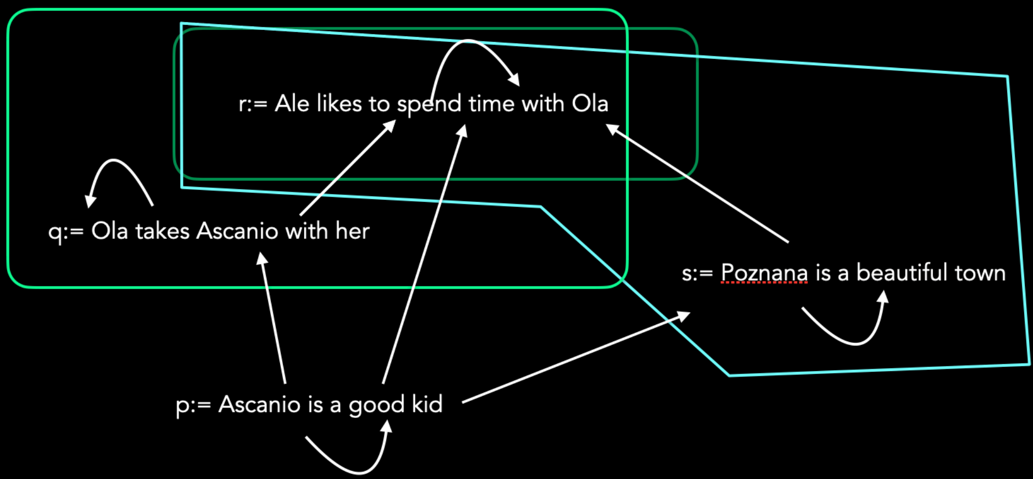

So what to do?

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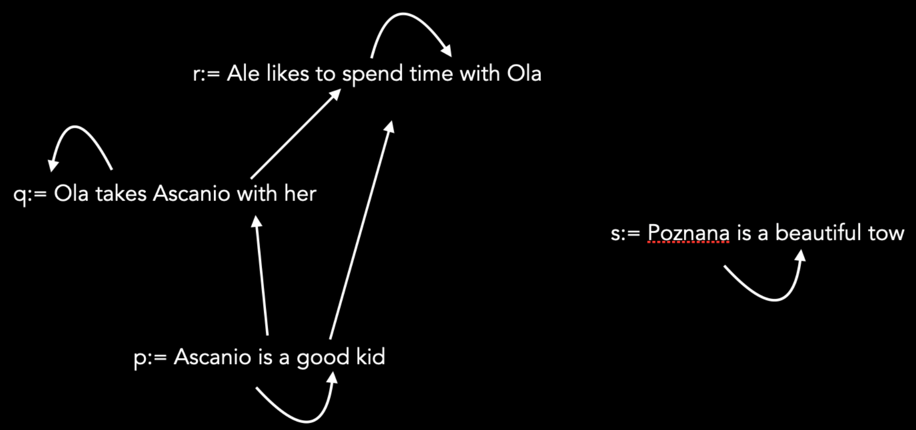

r:= Ale likes to spend time with Ola

q:= Ola takes Ascanio with her

s:= Poznana is a beautiful town

p:= Ascanio is a good kid

"Thing-thin … s theorem

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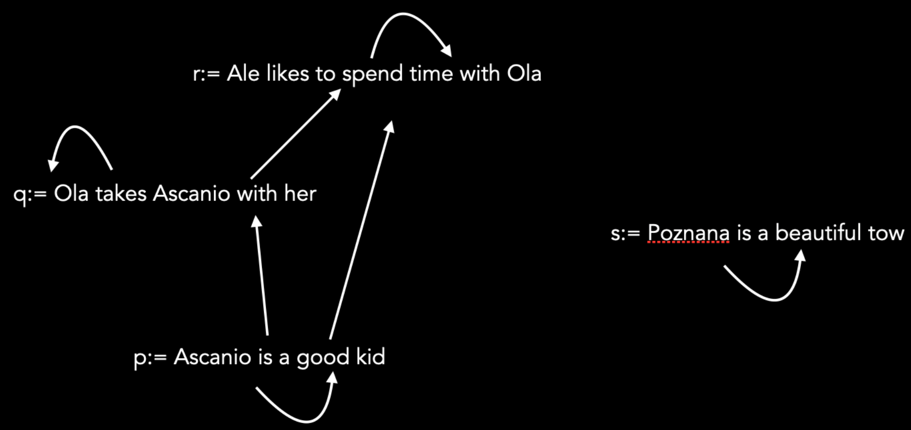

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"Thing-thin[...]" [...] s theorem

So what to do? Idea: I can define a valuation as the upset of the quasi-order generated by a point (aka principal cones)

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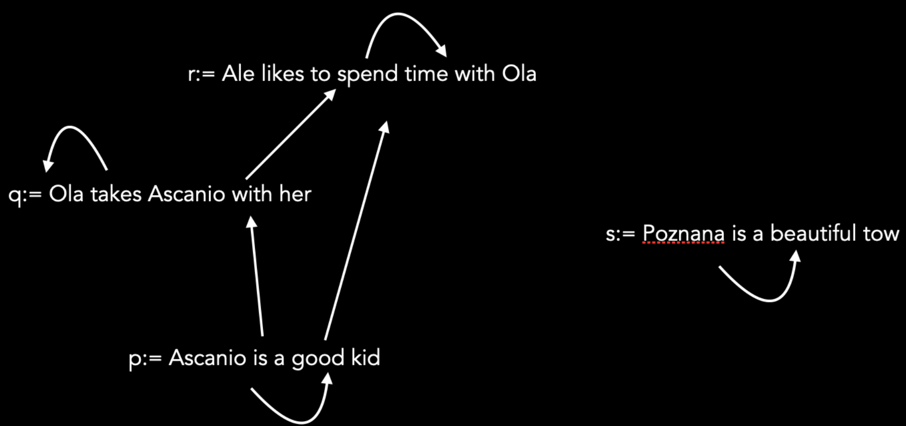

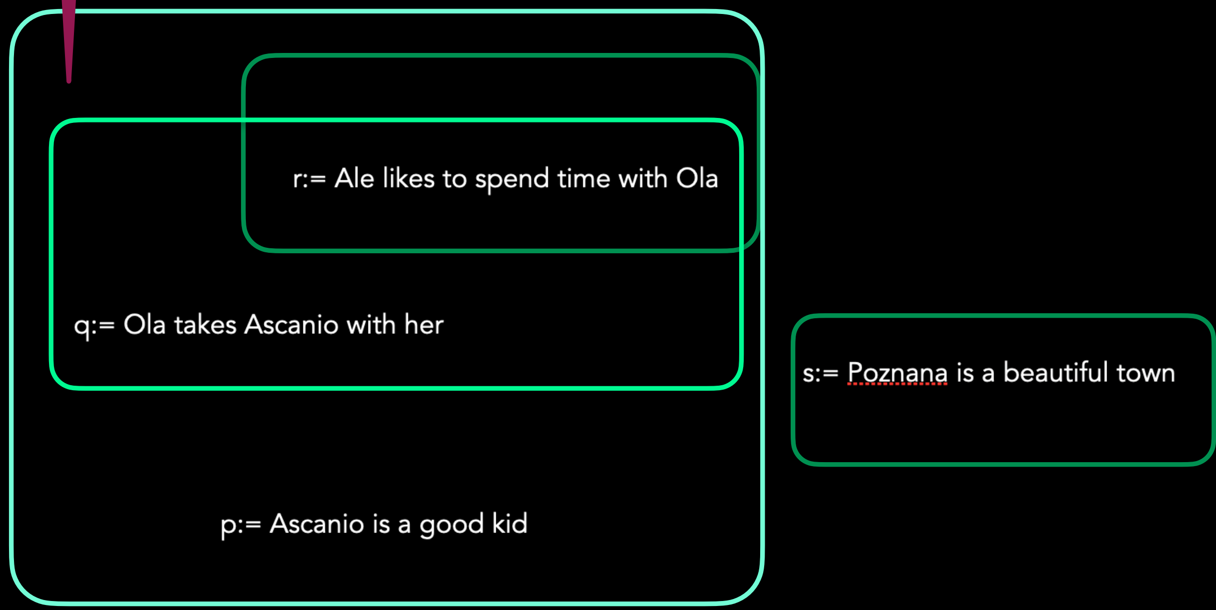

"Thing-thing" consequence relation: an abstract completeness theorem

- *Question*: Given a binary implication $\vdash$, is there a class of possible words $\mathfrak{S}$ over A whose semantic relation $\vdash_{\mathfrak{S}}$ coincide with the relation $\vdash$?

- *Theorem (abstract completeness for binary consequence relation)*: Given a quasi-order $\vdash$ over A it is always possible to find a class of possible words $\mathfrak{S}$ over A such that $\vdash = \vdash_{\mathfrak{S}}$.

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Proof: Consider the collection of all principal cones of A, i.e. $\mathfrak{S}(\vdash) := \{[a) := \{b \in A \mid a \vdash b\} \mid a \in A\}$.
Clearly the map $h : a \mapsto \{C \in \mathfrak{S}(\vdash) \mid a \in C\}$ is such that $a \vdash b$ iff $h(a) \subseteq h(b)$, meaning that $\vdash = \vdash_{\mathfrak{S}(\vdash)}$.

This provide a sort of representation of quasi-orders, that is a way of describing them “indirectly” via their principal cones; it shows how the abstract notion of pre-order can be “transformed” into a concrete relation defined using a collection of subsets.

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Fact: When $(A, \vdash)$ is a partially ordered set (poset), i.e. a pre-order which is also anti-symmetric, the map h is injective, and thus an embedding from $(A, \vdash)$ into $(\wp(\mathscr{C}(\vdash)), \subseteq)$.

"Thing-thing" consequence relation: absoluteness (categoricity)?

- Clearly it holds that $\mathfrak{G} \subseteq \mathfrak{G}(\vdash_{\mathfrak{G}})$
- *Question*: does the other direction holds too, and thus $\mathfrak{G}(\vdash_{\mathfrak{G}}) = \mathfrak{G}$? That is does a collection of possible worlds always characterise a unique pre-order?

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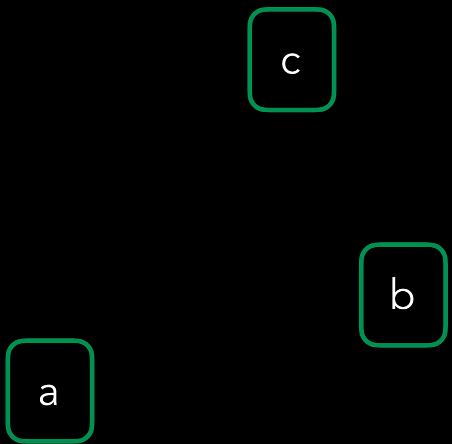

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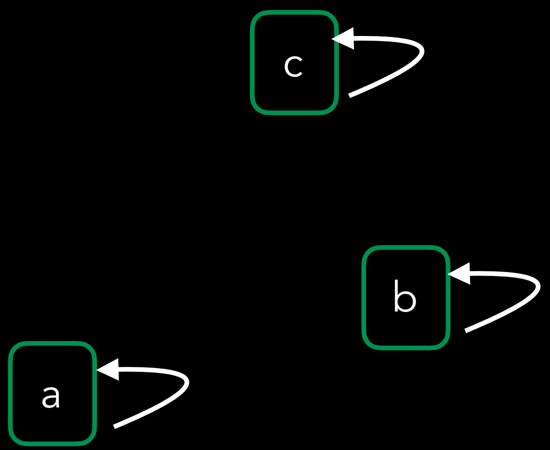

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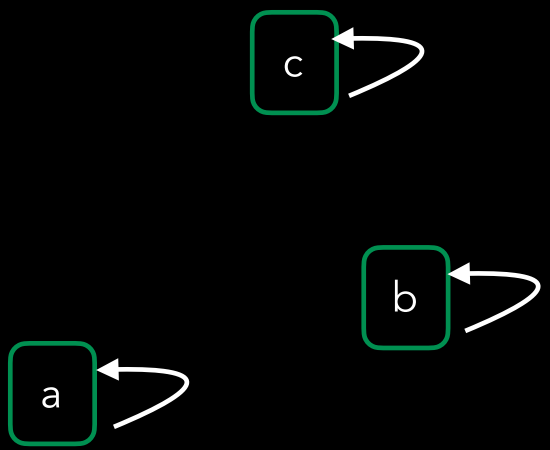

c

b

a

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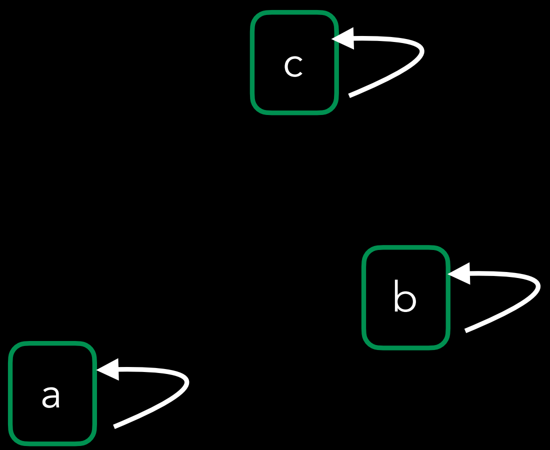

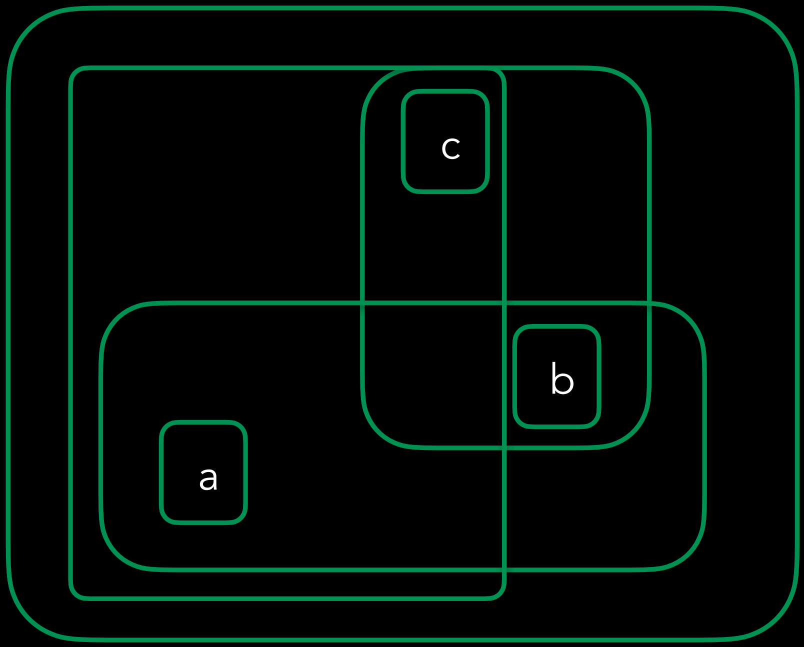

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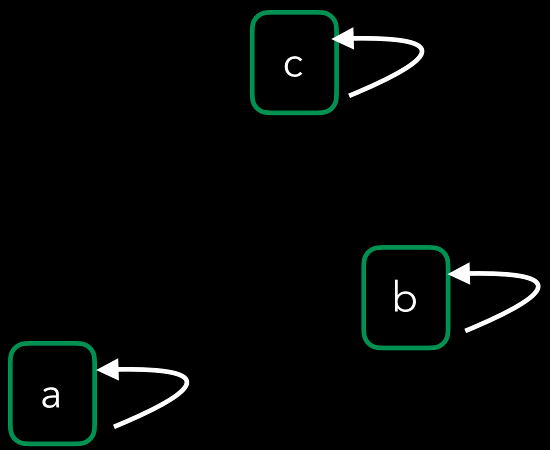

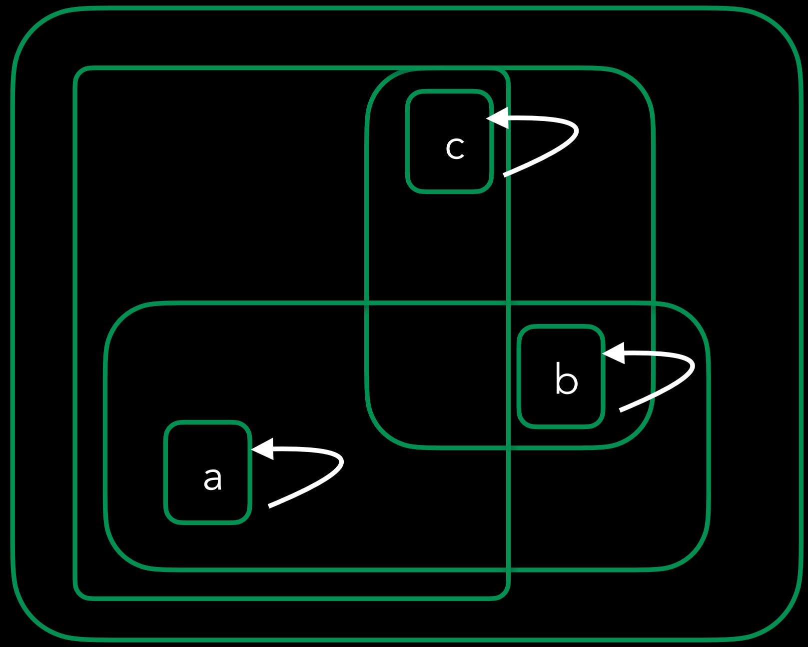

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  No. In fact, let $A = \{a, b, c\}$ and $\mathfrak{S} = \{\{a\}, \{b\}, \{c\}\}$. We have that $\mathfrak{S}(a) = \{\{a\}\}$, $\mathfrak{S}(b) = \{\{b\}\}$, and thus $\mathfrak{S}(x) = \{\{x\}\}$, for each $x \in A$. Clearly $x \vdash_{\mathfrak{S}} y$ iff $x = y$. Now, let $\mathfrak{S}' := \wp(A)$. It holds that $\vdash_{\mathfrak{S}} = \vdash_{\mathfrak{S}'}$.

- Generally speaking, a (abstract) logic does not need to have a unique semantics, as it may constitute the inferential basis for many different theories.

The idea of (abstract) asymmetric consequence relation

- The concept of (abstract asymmetric) **consequence relation**

$$\Gamma \vdash \varphi$$

"The set of hypotheses / assessments / background knowledge Γ entails φ"

"The set of things denoted by Γ entails entails the thing denoted by φ"

This view is proper to the “polish” tradition, stemming from the work by Tarski and Lindembaum, and later in the work of Łoś, Suszko, Wójcicki, and Czelakowski among others.

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- Analogous concept, that of **consequence operator**

$$\varphi \in \mathrm{Cn}(\Gamma)$$

“φ belongs to the set of propositions / things entailed by the set of hypotheses / things Γ”

- What kind of **properties** satisfies “⊢” (“Cn”) to be called a consequence relation (operator) ?

What is a (abstract) asymmetric consequence relation

- Let $\mathscr{L}$ be some set (of propositions, things, …)
- We say that a relation $\vdash\, \subseteq \wp(\mathscr{L}) \times \mathscr{L}$ is a (abstract asymmetric) **consequence relation** over $\mathscr{L}$, whenever it satisfies the following, for every $\varphi, \psi, \Gamma, \Delta$:
  - $\varphi \vdash \varphi$ (Reflexivity)
  - If $\Gamma \vdash \varphi$, for all $\varphi \in \Delta$, and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$ (Transitivity / Cut)

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What is a (abstract) asymmetric consequence r

Other "variant" of reflexivity implied by these two conditions:

- $\Gamma \vdash \varphi$, whenever $\varphi \in \Gamma$

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It is easy to check that with this reformulation, (Diluation) follows from (Reflexivity + Cut)

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- A consequence relation is called **finitary** if, for every φ, Γ:
  - If $\Gamma \vdash \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$ (Finitariness)

What is a (abstract) asymmetric consequence relation

- *Definition*: Let $\mathscr{L}$ be some set and $\vdash \subseteq \wp(\mathscr{L}) \times \mathscr{L}$ a relation. Whenever $\vdash$ is a (asymmetric) consequence relation, the pair $(\mathscr{L}, \vdash)$ is called a (abstract) asymmetric **consequence system,** sometimes also referred to as (abstract) deductive system, or Tarski structure.

Asymmetric consequence relation vs consequence operator

- Let $\mathscr{L}$ be some set (of propositions, things, …).
- A closure operator Cn : $\wp(\mathscr{L}) \to \wp(\mathscr{L})$ over the powerset $\wp(\mathscr{L})$ is often called an (abstract) **consequence operator** over $\mathscr{L}$; remember that as such it satisfies the following, for every Γ, Δ:

  - $\Gamma \subseteq \mathrm{Cn}(\Gamma)$ (Reflexivity)
  - If $\Gamma \subseteq \Delta$, then $\mathrm{Cn}(\Gamma) \subseteq \mathrm{Cn}(\Delta)$ (Monotonicity)
  - $\mathrm{Cn}(\mathrm{Cn}(\Gamma)) \subseteq \mathrm{Cn}(\Gamma)$ (Transitivity)

- A consequence operator over $\mathscr{L}$ is called **finitary** if, for every Γ:
  - $\mathrm{Cn}(\Gamma) = \bigcup_{\Delta \in \wp_\omega(\mathscr{L})} \mathrm{Cn}(\Delta)$ (Finitariness)

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Asymmetric consequence relation vs consequence operator

- *Fact*: Let $\mathscr{L}$ be some set. Then
  - If $\vdash \subseteq \wp(\mathscr{L}) \times \mathscr{L}$ is a (finitary) consequence relation, then the operator defined by $\mathsf{Cn}(\Gamma) := \{\varphi \in \mathscr{L} \mid \Gamma \vdash \varphi\}$ is a (finitary) consequence operator
  - If $\mathsf{Cn} : \wp(\mathscr{L}) \to \wp(\mathscr{L})$ is a (finitary) consequence operator, then the relation defined by ($\Gamma \vdash \varphi$ iff $\varphi \in \mathsf{Cn}(\Gamma)$) is a (finitary) consequence relation

Since we actually move freely between Cn and $\vdash$, we may also refer to $(\mathscr{L}, \mathsf{Cn})$ as a consequence system.

"Set (of things)-thing" consequence relation: an abstract completeness theorem

Remember that given a set of possible worlds $\mathfrak{S}$, we define $\mathfrak{S}(\varphi) := \{\mathfrak{s} \in \mathfrak{S} \mid \varphi \in \mathfrak{s}\}$. Similarly, we define $\mathfrak{S}(\Gamma) := \bigcap_{\gamma \in \Gamma} \mathfrak{S}(\gamma) = \{\mathfrak{s} \in \mathfrak{S} \mid \Gamma \subseteq \mathfrak{s}\}$, for a set $\Gamma \subseteq \mathscr{L}$

- *Definition*: The (semantic) asymmetric consequence relation $\vdash_{\mathfrak{S}}$ generated by $\mathfrak{S}$ over $\mathscr{L}$ is defined as

$$\Gamma \vdash_{\mathfrak{S}} \varphi \text{ if and only if } \mathfrak{S}(\Gamma) \subseteq \mathfrak{S}(\varphi)$$

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"Set (of things)-thing" consequence relation: an abstract completeness theorem

Let us consider a specific case over p,q,r, and with the following valuations.

	p	q	p,q

“Set (of things)-thing” consequence relation: an abstract completeness theorem

r	p	q, r	p,q, r

Properties?

"Set (of things)-thing" consequence relation: an abstract completeness theorem

r	p	q, r	p,q,r

We have that the consequence $\vdash_{\mathfrak{C}}$ generated by these four valuations is such that

- $q \vdash r$,
- $p, q \vdash r$ and $p, r \vdash q$
- $p \nvdash r$, and $r \nvdash q$ and $r \nvdash p$ and $q, r \nvdash p$

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actually r means "if p then q", in the classical sense

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For an arbitrary consequence system, is it possible to find a class of possible words / valuations inducing the same consequence relation?

"Set *(of things)-thing*" *consequence relation*: an abstract completeness theorem

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In order to prove it, we introduce the notion of closure operator on posets, and of closure systems.

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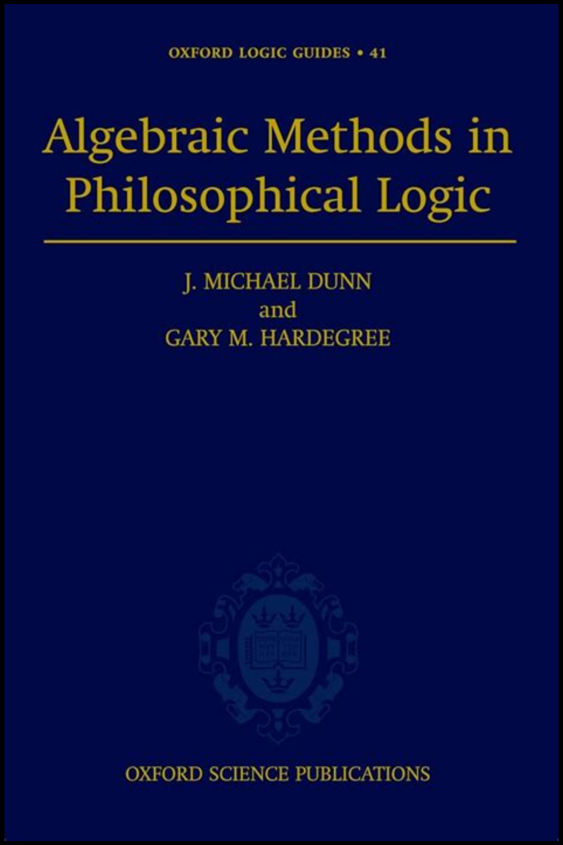
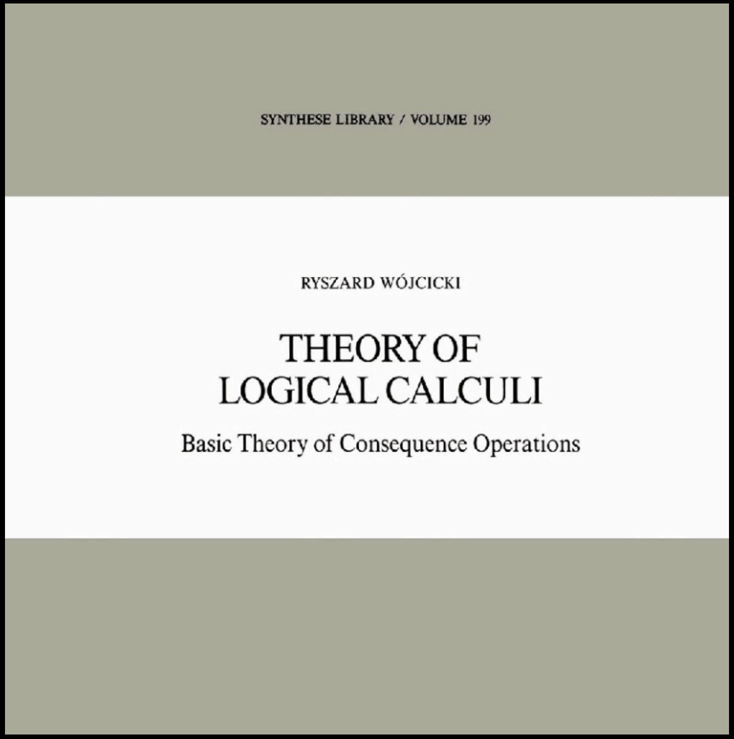
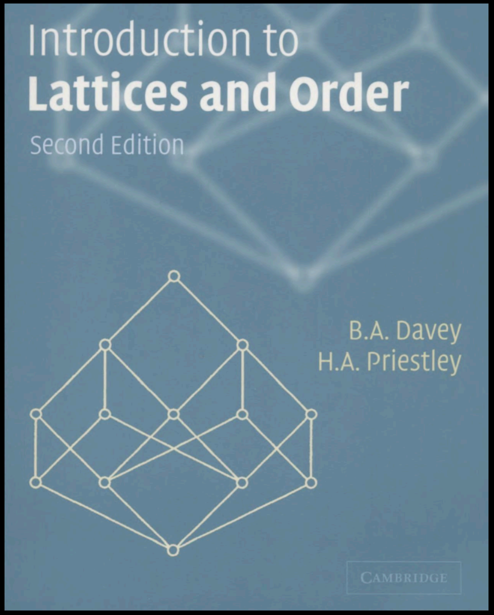
An first excursus on posets and lattices

Basics of the abstract (algebraic) view on logic: where







Partial orders

- *Definition*: Let A be a set and $\leq$ be a binary relation over A. The relation $\leq$ is **partial order** when it is a pre-order that is also anti-symmetric, that is whenever it satisfies the following, for every $a, b, c \in A$ it satisfies
  - $a \leq a$ (reflexivity)
  - If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)
  - If $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetry)

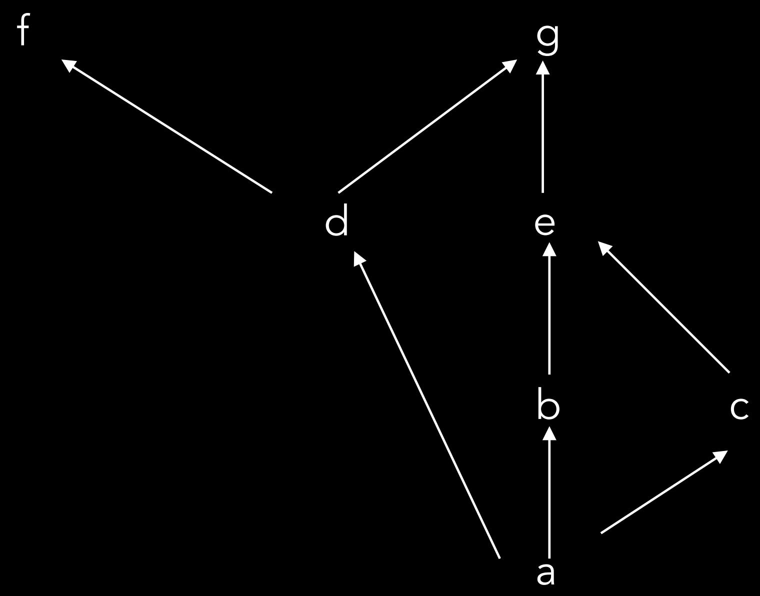
  If $\leq$ is a partial-order, the relational structure $(A, \leq)$ is called a **partially ordered set**, or poset.

  If $a \leq b$ but $b \neq a$, we thus write $a < b$.

Supremum / infimum on orders

- Given a poset $(A, \leq)$ and a subset $B \subseteq A$, an element $a \in A$ is said to be
  - the least upper bound, or **supremum**, of B if $b \leq a$, for every $b \in B$, and whenever there is $c \in A$ such that $b \leq c$, for every $b \in B$, then $a \leq c$; if a supremum of B exists, it is unique and is denoted by $\bigvee B$,
  - the greatest lower bound, or **infimum**, of B if $a \leq b$, for every $b \in B$, and whenever there is $c \in A$ such that $c \leq b$, for every $b \in B$, then $c \leq a$; if a infimum of B exists, it is unique and is denoted by $\bigwedge B$.

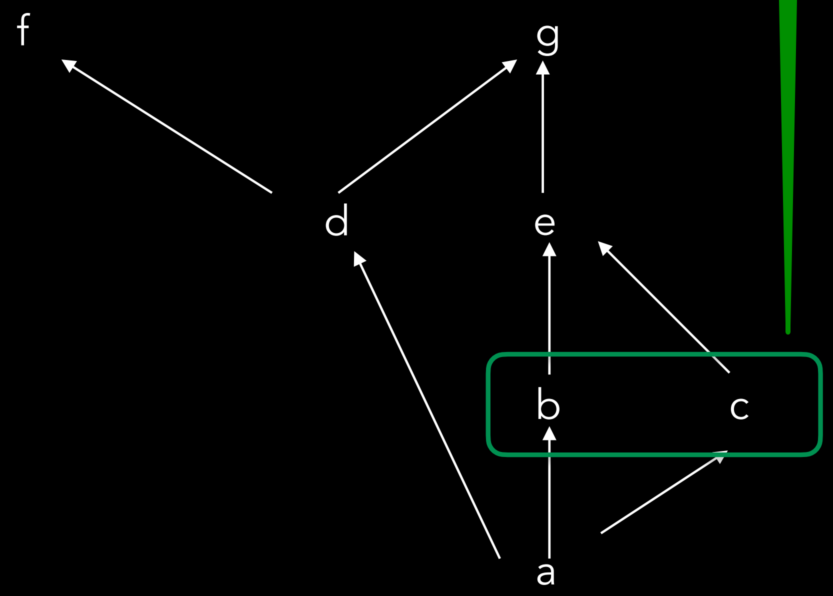
Supremum / infimum on orders



Supremum / infimum on orders

$\inf\{b, c\} = a$

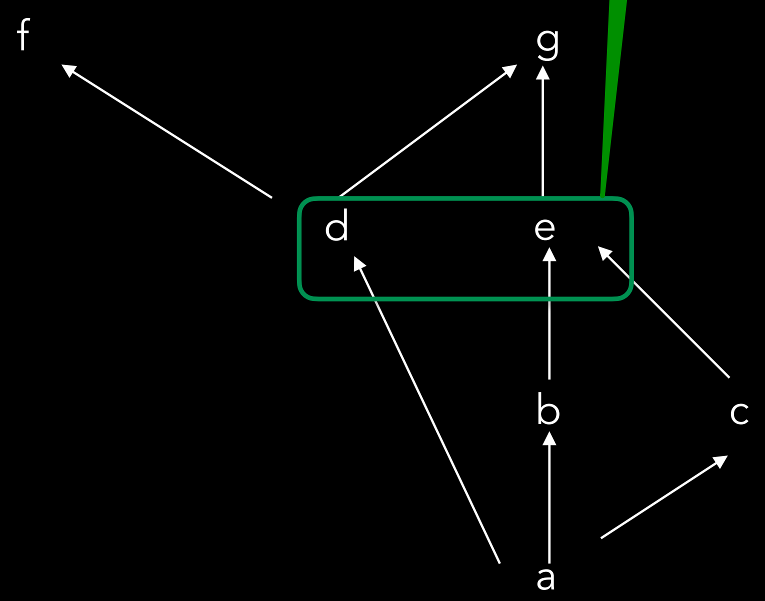
$\sup\{b, c\} = e$



Supremum / infimum on orders

$\inf\{d, e\} = a$

$\sup\{d, e\} = g$



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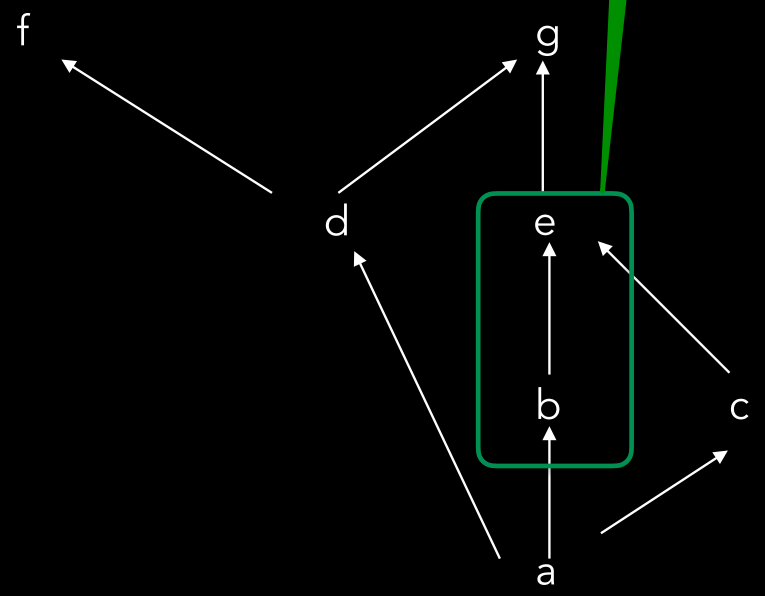
Supremum / infimum on orders

$\inf\{b, e\} = b$

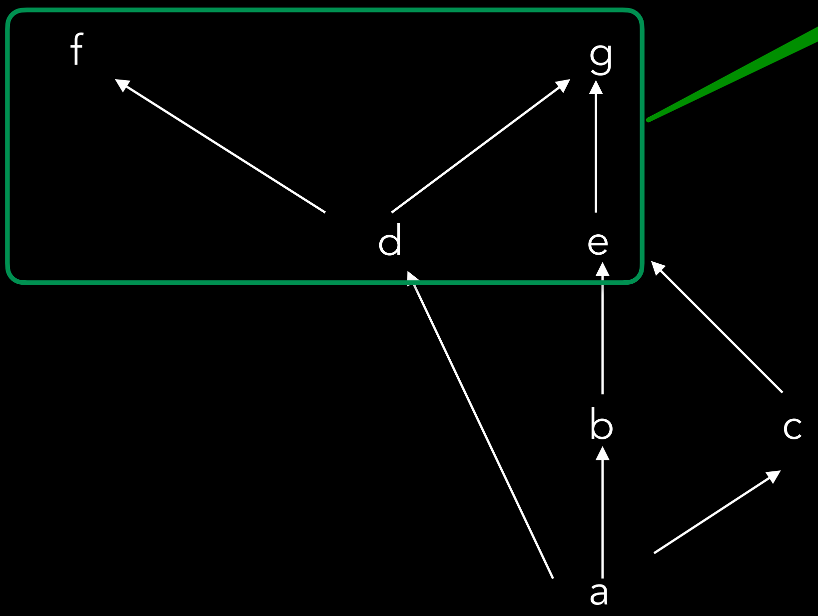
$\sup\{b, e\} = e$



Supremum / infimum on orders

$\inf\{d, e, f, g\} = a$

$\sup\{d, e, f, g\} = ?$



Orders and lattices

- *Definition*: Let $(A, \leq)$ be a poset. If it is such that
  - every subset of type $\{a, b\}$ has a supremum $a \vee b$ (also called join), then it is called a **join-semi-lattice**
  - every subset of type $\{a, b\}$has a infimum $a \wedge b$ (also called meet), then it is called a **meet-semi lattice**
  - every subset of type $\{a, b\}$ has both a supremum and an infimum, is it called a **lattice.**

- *Example*: the poset $(\wp(A), \subseteq)$ has both joins (set-theoretic union) and meets (set-theoretic intersection), and it is thus (obviously) a lattice (as we are going to see it naturally gives rise to the usual algebraic lattice $(\wp(A), \cap, \cup)$).

Algebraic lattices

- *Definition*: An algebra $(A, \wedge, \vee)$ is an algebraic **lattice** whenever the two operations on A are both commutative and associative, and they satisfies the absorptions laws
  - $x \wedge (x \vee y) = x,$
  - $x \vee (x \wedge y) = x,$

  meaning in particular that they also satisfies the following idempotent laws
  - $x \wedge x = x,$
  - $x \vee x = x.$

- *Example*: Consider a set A. Then the powerset algebra $(\wp(A), \cap, \cup)$ is a algebraic lattice.

Lattices and orders

- *Fact*: If $(A, \wedge, \vee)$ is an algebraic lattice, then we can define a poset $(A, \leq)$ which is a lattice by setting, for every $a, b \in A$

$$a \leq b \text{ iff } a \vee b = b$$

Or equivalently, by absorption, by setting $a \leq b$ iff $a \wedge b = a$.

- *Example*: Consider the lattice $(\wp(A), \cap, \cup)$. The the partial order $a \leq b$ iff $a \cup b = b$ is simply the usual subset relation.

- The way of viewing lattices as partially ordered sets, with certain additional properties, is extremely useful from a logical point of view.

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Semi-lattices and orders

- *Definition*: An algebra $(A, \circ)$ is an algebraic **semi-lattice** whenever the binary operation $\circ$ on A is both commutative and associative, and it satisfies the idempotent law
  - $x \circ x = x$

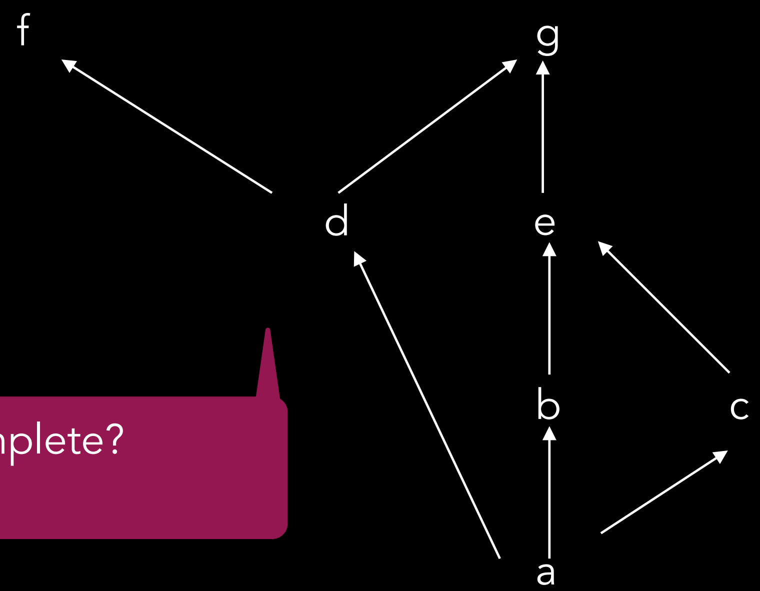
- *Fact*: As for algebraic lattices , any algebraic semi-lattice $(A, \circ)$ gives rise to a join-semi lattice $(A, \leq)$ by setting either $a \leq b$ iff $a \circ b = b$ (and thus $\circ$ is seen as a join / disjunction) or to a meet-semi lattice $(A, \leq)$ by setting $a \leq b$ iff $a \circ b = a$ (and thus $\circ$ is seen as a meet / conjunction)

Bounded and complete lattices

- *Definition*: Let $(A, \leq)$ be a poset that is a lattice.
  - Whenever both $0_A := \bigwedge A$ and $1_A := \bigvee A$ exist, we say that the poset (lattice) is **bounded,**
  - Whenever $1_A \in A$ and $\bigwedge B$ exists for every $B \subseteq A$ (with $1_A = \bigwedge \varnothing$), we say that the lattice is a **closure system**,
  - Whenever both $\bigwedge B$ and $\bigvee B$ exist for every $B \subseteq A$ (with $1_A = \bigwedge \varnothing$ and $0_A = \bigvee \varnothing$), we say that the lattice is **complete**.

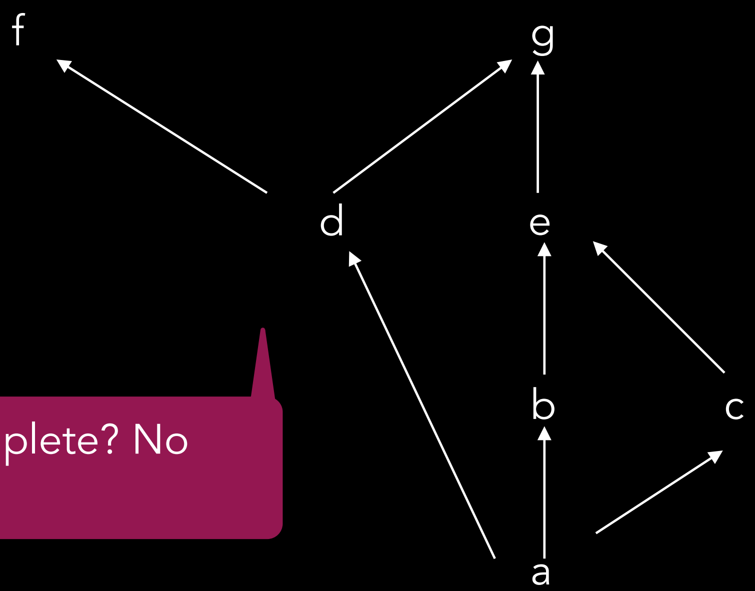
- *Example*: the powerset lattice $(\wp(\mathscr{L}), \subseteq)$ is obviously complete. Notice however that not every lattice is bounded, and thus a fortiori complete, e.g. $(\mathbb{Z}, \leq)$.

Bounded and complete lattices



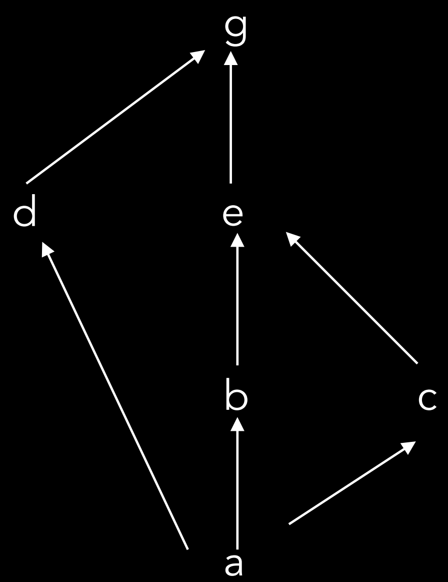
bounded? complete?

Bounded and complete lattices



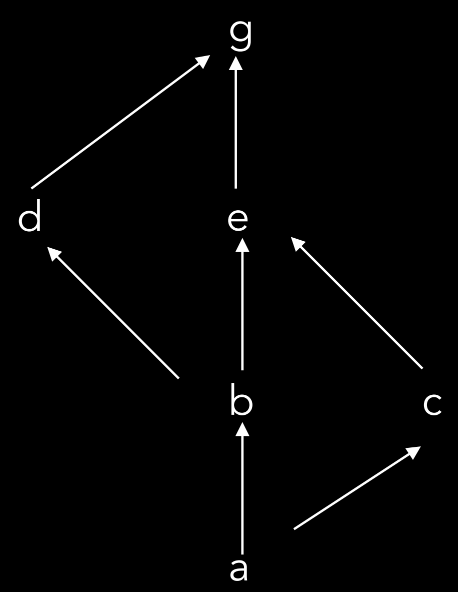
bounded? complete? No

Bounded and complete lattices



But this is bounded and complete.

Bounded and complete lattices



and also this is bounded anc complete.

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Lattices, orders and closure operators

- *Definition*: Let $(A, \leq)$ be a poset that is a lattice. A function Cl : $A \to A$ is called a **closure operator** on A whenever it satisfies the following, for every $a, b \in A$:

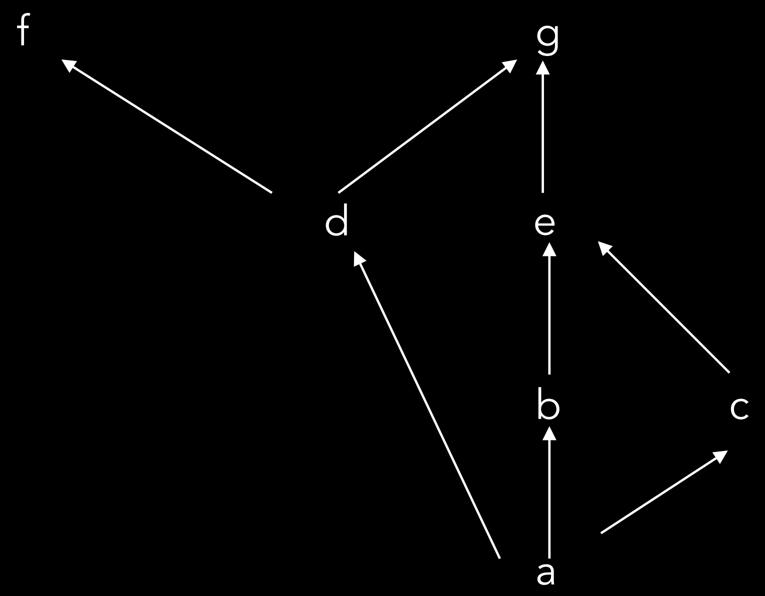
  (C1) $a \leq \mathrm{Cl}(a)$

  (C2) If $a \leq b$, then $\mathrm{Cl}(a) \leq \mathrm{Cl}(b)$

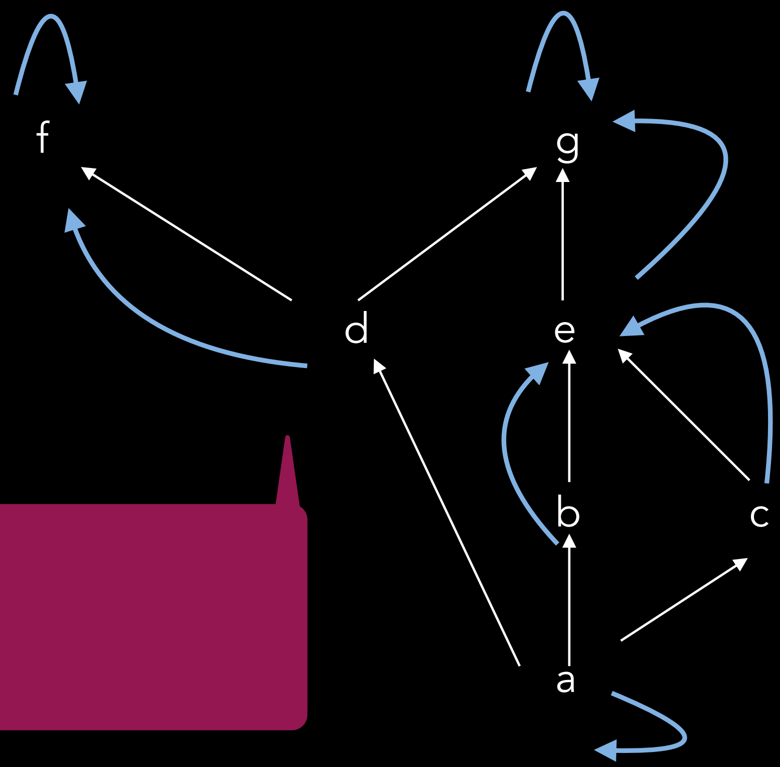
  (C3) $\mathrm{Cl}(\mathrm{Cl}(a)) \leq \mathrm{Cl}(a)$

  Elements $a \in A$ such that $a = \mathrm{Cl}(a)$ are called **closed**. The collection of all closed elements of A is denoted by $\mathfrak{C}_A$, or simply $\mathfrak{C}$ when the underlying poset is clear.

Lattices, orders and closure operators

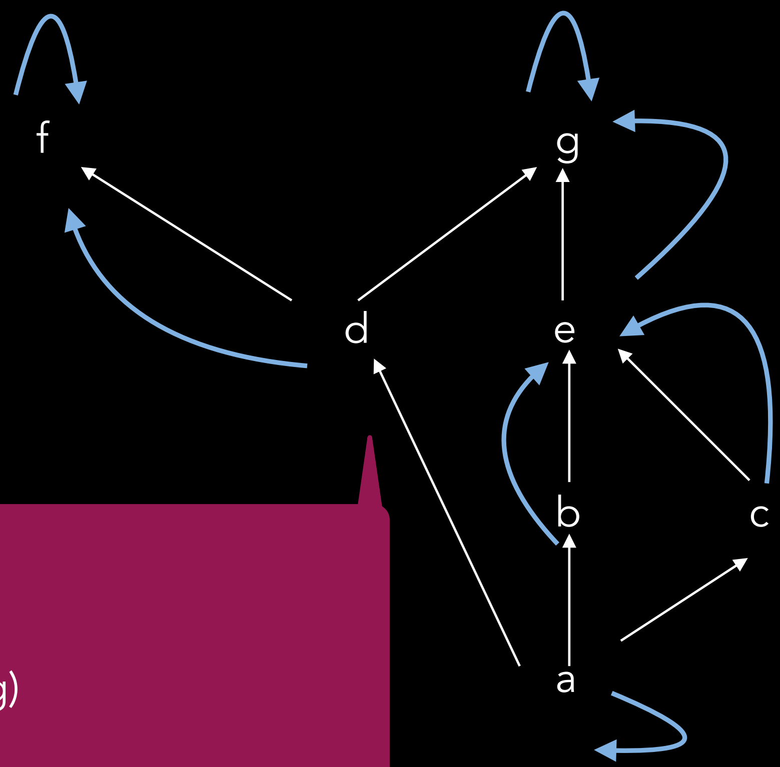


Lattices, orders and closure operators



working?

Lattices, orders and closure operators



working? no

Cl(e) > e=Cl(b)

d < g but Cl(d) ≮ Cl(g)

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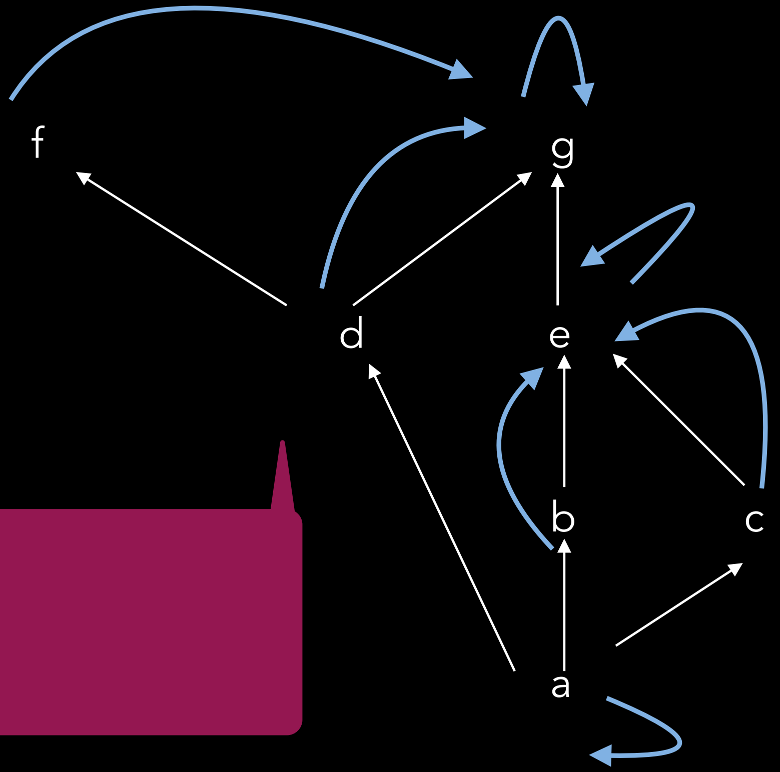
Lattices, orders and closure operators



working?

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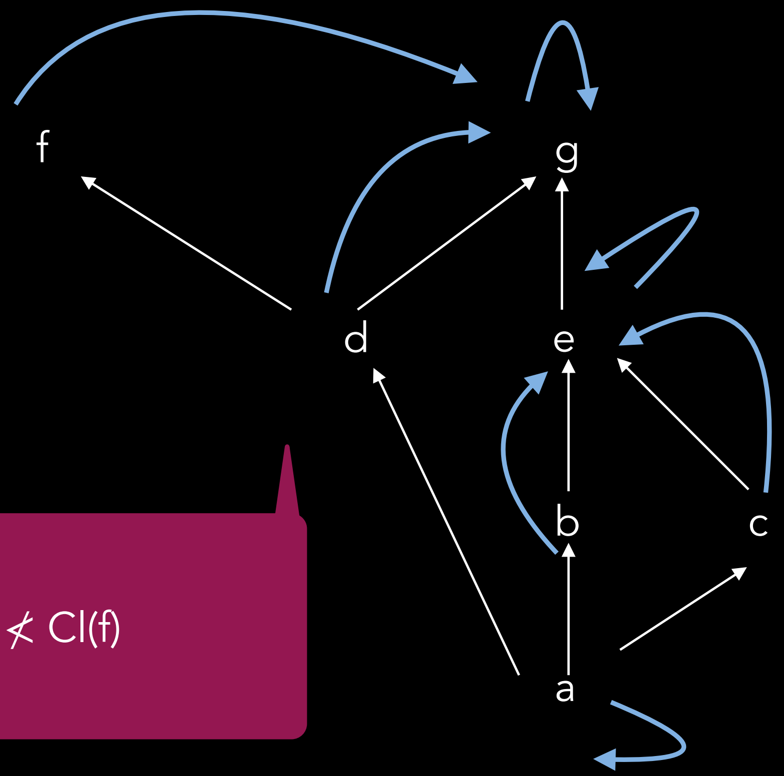
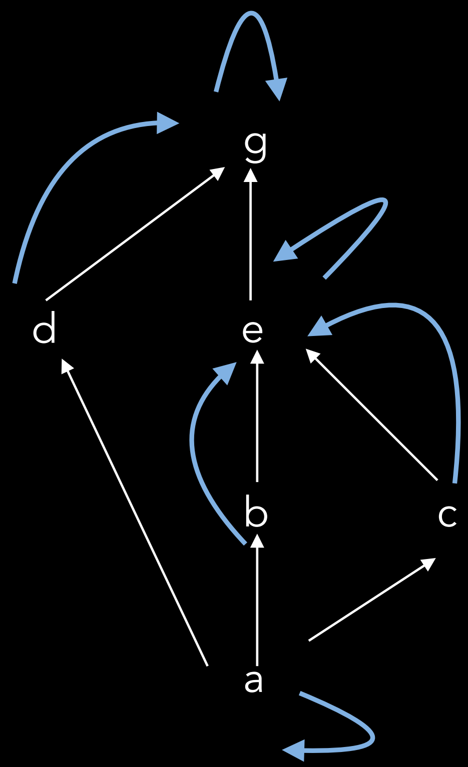
Lattices, orders and closure operators



working?

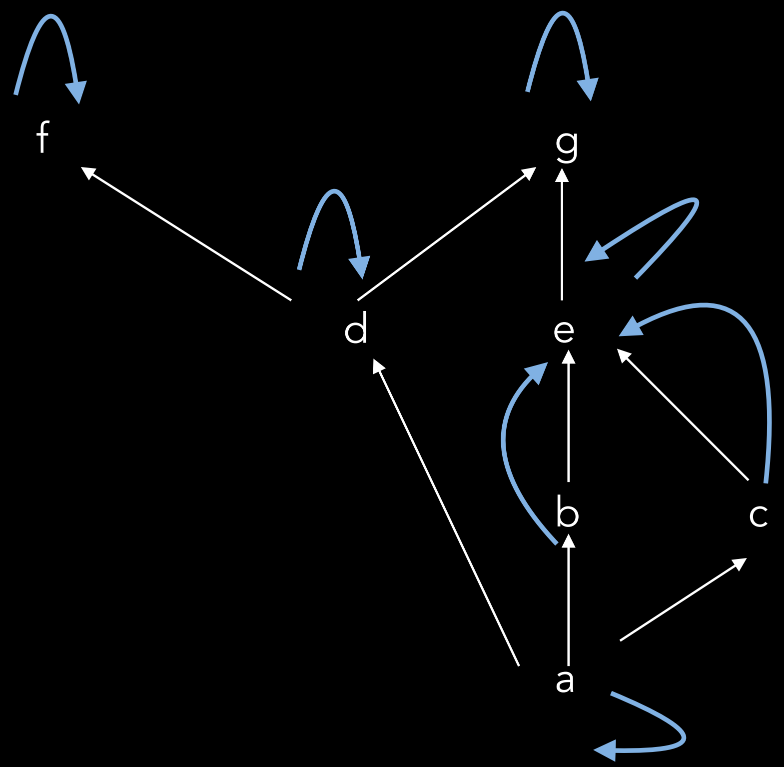
no, because $f \not\leq Cl(f)$

Lattices, orders and closure operators



If f would not be there, this is ok

Lattices, orders and closure operators



Lattices, orders and closure operators

- *Definition*: Let $(A, \leq)$ be a poset that is a lattice. A function $\mathsf{Cl} : A \to A$ is called a **closure operator** on A whenever it satisfies the following, for every $a, b \in A$:

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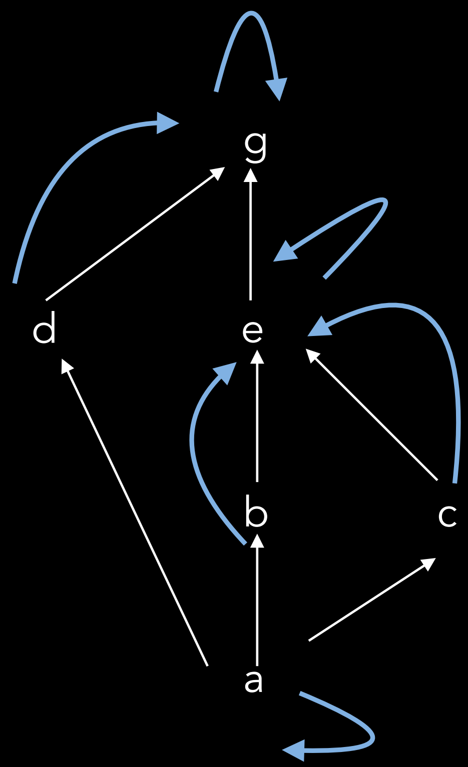
  Elements $a \in A$ such that $a = \mathsf{Cl}(a)$ are called **closed**. The collection of all closed elements of A is denoted by $\mathfrak{C}_A$, or simply $\mathfrak{C}$ when the underlying poset is clear.

Let $(\mathscr{L}, \vdash)$ be consequence system, and consider the complete powerset lattice $(\wp(\mathscr{L}), \subseteq)$. Then the consequence operator Cn is, by definition, a closure operator on $\wp(\mathscr{L})$. Closed sets $\Gamma = \mathsf{Cn}(\Gamma)$ are usually called "theories" and their collection denoted by $\mathfrak{T}(\vdash)$ or $\mathfrak{T}_{\vdash}$.

Lattices, orders and closure operators

- *Theorem*: Let $(A, \leq)$ be a poset that is a lattice, and $\mathrm{Cl} : A \to A$ be a closure operator on A
    1. $\mathrm{Cl}(a) = \bigwedge \{b \in \mathfrak{C}_A \mid a \leq b\}$
    2. If $(A, \leq)$ is complete, then $(\mathfrak{C}_A, \leq)$ is also a complete lattice (and thus a closure system), and such that $\bigwedge_{\mathfrak{C}_A} \mathscr{A} = \bigwedge \mathscr{A}$ and $\bigvee_{\mathfrak{C}_A} \mathscr{A} = \mathrm{Cl}(\bigvee \mathscr{A})$, for every family $\mathscr{A} \subseteq \mathfrak{C}_A$.

Lattices, orders and closure operators



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Proof: For the first point, since $a \leq \mathsf{Cl}(a) = \mathsf{ClCl}(a)$, we have that $\mathsf{Cl}(a) \in \mathfrak{C}_A$. Now assume $a \leq b \leq \mathsf{Cl}(a)$, for some $b \in \mathfrak{C}_A$. Thus $\mathsf{Cl}(a) \leq \mathsf{Cl}(b) = b \leq \mathsf{ClCl}(a) = \mathsf{Cl}(a)$, meaning $\mathsf{Cl}(a) = \bigwedge \{b \in \mathfrak{C}_A \mid a \leq b\}$. For the second point, we first verify that $\mathsf{Cl}(\bigwedge \mathscr{A}) \leq a = \mathsf{Cl}(a)$, for every $a \in \mathscr{A}$, meaning that $\mathsf{Cl}(\bigwedge \mathscr{A}) \leq \bigwedge \mathscr{A}$, and thus by (C1) we get $\bigwedge \mathscr{A} \in \mathfrak{C}_A$. Then by definition and the previous points:

$$\bigvee_{\mathfrak{C}_A} \mathscr{A} = \bigwedge \{b \in \mathfrak{C}_A \mid a \leq b, \forall a \in \mathscr{A}\} = \bigwedge \{b \in \mathfrak{C}_A \mid \bigvee \mathscr{A} \leq b\} = \mathsf{Cl}(\bigvee \mathscr{A}).$$

Lattices, orders and closure operators

By the first point of the previous theorem, we immediately get that:

- *Corollary*: Let $(A, \leq)$ be a complete lattice, and $\mathrm{Cl}_i : A \to A$ be some closure operators, with $i = 1,2$. Thus $\mathfrak{C}_1 = \mathfrak{C}_2$ implies that $\mathrm{Cl}_1 = \mathrm{Cl}_2$.

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Corollaries: Let $(\mathscr{L}, \vdash)$ be consequence system.

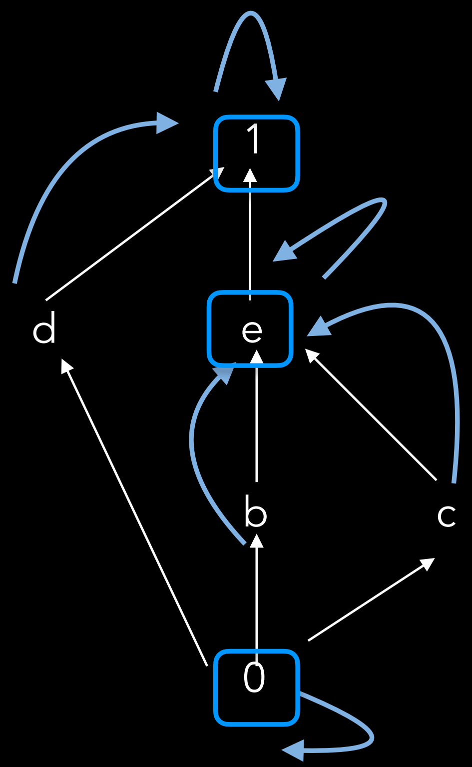
1. $(\mathfrak{T}_\vdash, \subseteq)$ is a complete lattice (and thus a closure system), and such that $\bigwedge_{\mathfrak{T}_\vdash} \mathscr{C} = \bigcap \mathscr{C}$ and $\bigvee_{\mathfrak{T}_\vdash} \mathscr{C} = \mathsf{Cn}(\bigvee \mathscr{C})$, for every family of theories $\mathscr{C} \subseteq \mathfrak{T}_\vdash$.
2. Given a consequence system $(\mathscr{L}, \vdash')$, $\mathfrak{T}_\vdash = \mathfrak{T}_{\vdash'}$ implies that $\vdash = \vdash'$.

Lattices, orders and closure operators

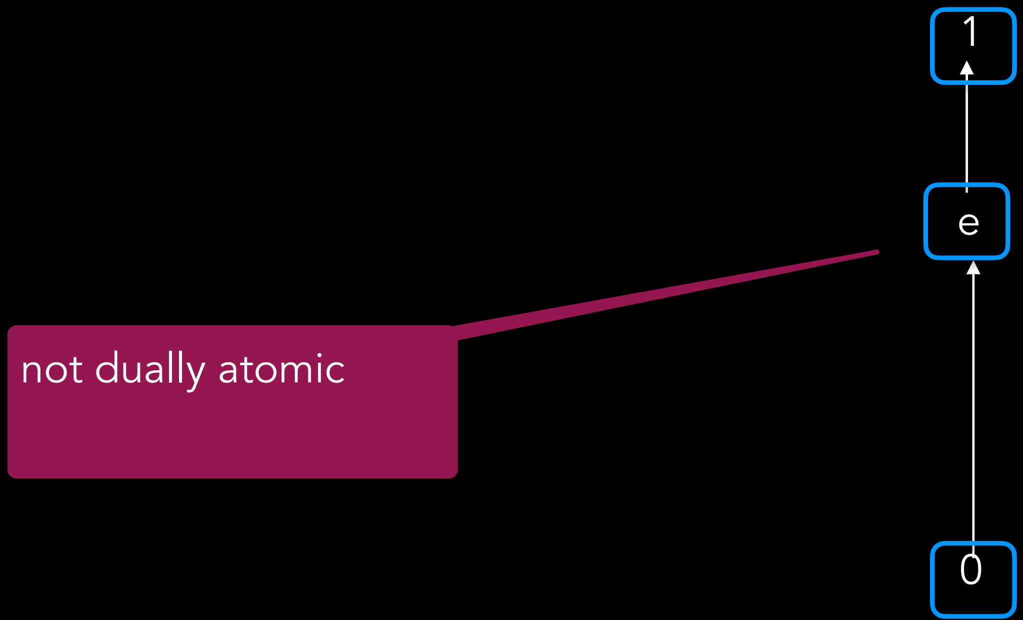
By both points of the previous theorem, in some special cases we can actually obtain a nice, simple characterisation of the closure operator: in term of some kind of maximal closed elements.

- *Definition*: Let $(A, \leq)$ be a bounded lattice. An element $a \in A$, is an atom if $0 < a$, and there is no $a \neq b \in A$ such that $0 < b \leq a$, and a dual atom if $a < 1$, and there is no $a \neq b \in A$ such that $a \leq b < 1$. $(A, \leq)$ is thus said to be atomic if for every $a \in A \backslash \{0\}$ there is a set of atoms B such that $b = \bigvee A$, and dually atomic if for every $a \in A \backslash \{1\}$ there is a set of dual atoms B such that $b = \bigwedge A$.

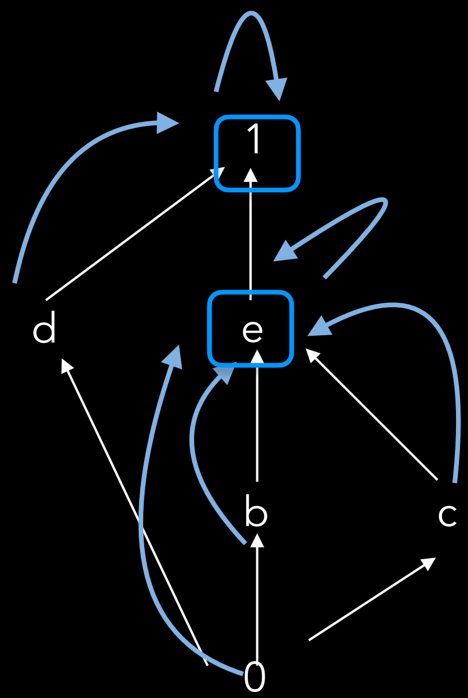
Lattices, orders and closure operators



Lattices, orders and closure operators



Lattices, orders and closure operators



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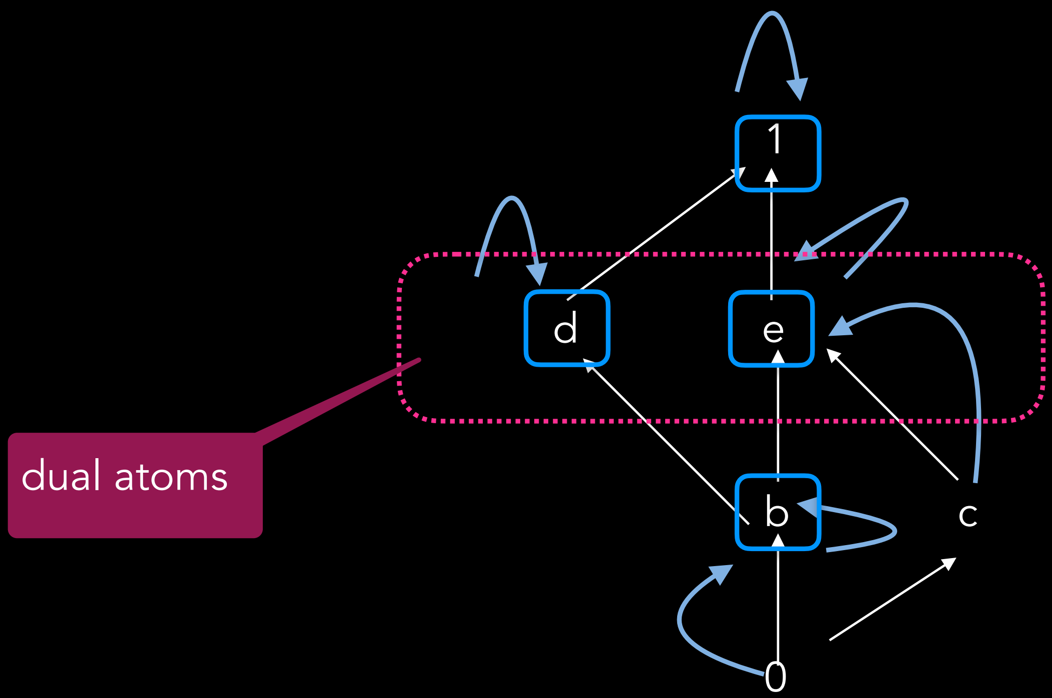

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- Corollary: Let $(A, \leq)$ be a complete lattice, and Cl $: A \to A$ be a closure operator. Assume $(\mathfrak{C}_A, \leq)$ is dually atomic. Then Cl$(a) = \bigwedge \{b \in \mathfrak{M}_A \mid a \leq b\}$, where $\mathfrak{M}_A$ is the collection of all dual atoms of $(\mathfrak{C}_A, \leq)$.

Closure systems and closure operators

We now verify that closure systems and closure operators (on complete lattices) are two faces of the same coin. In fact it is immediate to verify that

- *Theorem*: Let $(A, \leq)$ be a complete lattice, and consider $B \subseteq A$ such that $(B, \leq)$ is a closure system. Then $\mathrm{Cl}_B : A \to A$ defined $\mathrm{Cl}_B(a) := \bigwedge \{b \in B \mid a \leq b\}$ is a closure operator and clearly $\mathfrak{C}_{\mathrm{Cl}_B} = B$. In particular, whenever $B = \mathfrak{C}_{\mathrm{Cl}'}$, for some closure operator $\mathrm{Cl}' : A \to A$, we have that $\mathrm{Cl}' = \mathrm{Cl}_B$.

- *Corollary*: Over complete lattices, there is a bijection between closure systems and closure operators.

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Let $(\mathscr{L}, \vdash)$ be consequence system. Remember that the corresponding consequence operator Cn given by $\mathsf{Cn}(\Gamma) := \{\varphi \in \mathscr{L} \mid \Gamma \vdash \varphi\}$ is a closure operator on $\wp(\mathscr{L})$ and thus $(\mathfrak{T}_\vdash, \subseteq)$ is a closure system. Hence, $\mathfrak{C}_{\mathsf{Cl}_{\mathfrak{T}_\vdash}} = \mathfrak{T}_\vdash$. But since $\mathfrak{C}_{\mathsf{Cl}_{\mathfrak{T}_\vdash}} = \mathfrak{C}_{\mathsf{Cn}} := \mathfrak{T}_\vdash$, we have that $\mathsf{Cl}_{\mathfrak{T}_\vdash} = \mathsf{Cn}$ and thus

$$\mathsf{Cn}(\Gamma) = \bigcap \{\Delta \in \mathfrak{T}_\vdash \mid \Gamma \subseteq \Delta\}.$$

Closure syste[...]

We now verify tha[...]
In fact it is immed[...]

In particular if $(\mathfrak{T}_{\vdash}, \subseteq)$ is dually atomic, and $\mathfrak{M}_{\vdash}$ is the collection of maximal "non trivial" (i.e. different from $\mathscr{L}$) theories of $(\mathscr{L}, \vdash)$, it holds that

$$\mathrm{Cn}(\Gamma) = \bigcap \{\Delta \in \mathfrak{M}_{\vdash} \mid \Gamma \subseteq \Delta\}.$$

- *Theorem*: Let $(A, \leq)$ be a complete lattice and consider $B \subseteq A$ such that $(B, \leq)$ is a closure system. Then $\mathrm{Cl}_B : A \to A$ defined $\mathrm{Cl}_B(a) := \bigwedge \{b \in B \mid a \leq b\}$ is a closure operator and clearly $\mathfrak{C}_{\mathrm{Cl}_B} = B$. In particular, whenever $B = \mathfrak{C}_{\mathrm{Cl}'}$, for some closure operator $\mathrm{Cl}' : A \to A$, we have that $\mathrm{Cl}' = \mathrm{Cl}_B$.

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Back to completeness for asymmetric consequence relation

Remember that a (semantic) asymmetric relation $\vdash_{\mathfrak{S}}$ generated by $\mathfrak{S}$ over $\mathscr{L}$ is defined as

$$\Gamma \vdash_{\mathfrak{S}} \varphi \text{ if and only if } \mathfrak{S}(\Gamma) \subseteq \mathfrak{S}(\varphi)\text{, where } \mathfrak{S}(\varphi) := \{\Delta \in \mathfrak{S} \mid \varphi \in \Delta\} \text{ and } \mathfrak{S}(\Gamma) := \{\Delta \in \mathfrak{S} \mid \Gamma \subseteq \Delta\}$$

- *Corollary (completeness for asymmetric consequence)*: Let $(\mathscr{L}, \vdash)$ be a consequence system. Then it is always possible to find a class of possible words $\mathfrak{S}$ over A such that $\vdash = \vdash_{\mathfrak{S}}$.

  Proof: Consider the collection $\mathfrak{T}_{\vdash}$ of all theories of $(\mathscr{L}, \vdash)$. It is enough to check that verify $\mathfrak{T}_{\vdash} = \mathfrak{T}_{\vdash_{\mathfrak{T}_{\vdash}}}$.

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Or, one could check that that for every pair (Γ, φ)

$$\bigcap \{\Delta \in \mathfrak{T}_{\vdash} \mid \varphi \in \Delta\} \subseteq \bigcap \{\Delta \in \mathfrak{T}_{\vdash} \mid \Gamma \subseteq \Delta\} \text{ if and only if } \{\Delta \in \mathfrak{T}_{\vdash} \mid \Gamma \subseteq \Delta\} \subseteq \{\Delta \in \mathfrak{T}_{\vdash} \mid \varphi \in \Delta\}$$

or stated otherwise, that

$$\bigcap \mathfrak{T}_{\vdash}(\varphi) \subseteq \bigcap \mathfrak{T}_{\vdash}(\Delta\} \text{ if and only if } \mathfrak{T}_{\vdash}(\Delta) \subseteq \mathfrak{T}_{\vdash}(\varphi)$$

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  Proof: Consider the collection $\mathfrak{T}_{\vdash}$ of all theories of $(\mathscr{L}, \vdash)$. It is enough to check that verify $\mathfrak{T}_{\vdash} = \mathfrak{T}_{\vdash_{\mathfrak{T}_{\vdash}}}$. Assume $\Delta \in \mathfrak{T}_{\vdash}$ and $\Delta \vdash_{\mathfrak{T}_{\vdash}} \varphi$. Notice that $\varphi \in \Delta$ since $\Delta \in \mathfrak{T}_{\vdash}(\Delta) \subseteq \mathfrak{T}_{\vdash}(\varphi)$, meaning that $\Delta \in \mathfrak{T}_{\vdash_{\mathfrak{T}_{\vdash}}}$. Now, assume that $\Delta \in \mathfrak{T}_{\vdash_{\mathfrak{T}_{\vdash}}}$, and consider $\varphi \in \mathscr{L}$ such that $\Delta \vdash \varphi$. This means that $\varphi \in \bigcap \{\Phi \in \mathfrak{T}_{\vdash} \mid \Delta \subseteq \Phi\} = \bigcap \mathfrak{T}_{\vdash}(\Delta)$. Hence if $\Phi \in \mathfrak{T}_{\vdash}(\Delta) := \{\Delta \in \mathfrak{S} \mid \Gamma \subseteq \Delta\}$, it holds that $\varphi \in \Phi$, and therefore $\Phi \in \mathfrak{T}_{\vdash}(\varphi)$, thus $\Delta \vdash_{\mathfrak{T}_{\vdash}} \varphi$. We thus have that $\varphi \in \Delta$, since $\Delta \in \mathfrak{T}_{\vdash_{\mathfrak{T}_{\vdash}}}$, meaning that $\Delta \in \mathfrak{T}_{\vdash}$.

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“*Set-thing*” *consequence relation*: absoluteness (categoricity)?

Again, there is no unique semantics for an asymmetric consequence relation.

To see this, set $\mathscr{L} := \{p, q, r\}$, and the two following classes

- $\mathfrak{S}_1 := \{\mathfrak{s} : \{p, q, r\} \to \{\mathsf{a}, \mathsf{r}\} \mid (\mathfrak{s}(p) = \mathsf{a} \leftrightarrow \mathfrak{s}(q) = \mathsf{r}) \wedge \mathfrak{s}(r) = \mathsf{a}\}$
- $\mathfrak{S}_2 := \{\mathfrak{s} : \{p, q, r\} \to \{\mathsf{a}, \mathsf{r}\} \mid \mathfrak{s}(r) = \mathsf{a}\}$

With respect to $\mathfrak{S}_1$ it is like $p \equiv \neg q$ and $r \equiv (q \vee \neg q)$

Clearly $\mathfrak{S}_1 \subsetneq \mathfrak{S}_2$. But it is easy to check that actually $\vdash_{\mathfrak{S}_1} = \vdash_{\mathfrak{S}_2}$.

"*Set-thing*" *consequence relation*: absoluteness (categoricity)?

Again, there is no unique semantics for an asymmetric consequence relation.

To see this, set $\mathscr{L} := \{p, q, r\}$, and the two following classes

- $\mathfrak{S}_1 := \{\mathfrak{s} : \{p, q, r\} \to \{\mathsf{a}, \mathsf{r}\} \mid (\mathfrak{s}(p) = \mathsf{a} \leftrightarrow \mathfrak{s}(q) = \mathsf{r}) \wedge \mathfrak{s}(r) = \mathsf{a}\}$
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Clearly $\mathfrak{S}_1 \subsetneq \mathfrak{S}_2$. But it is easy to check that actually $\vdash_{\mathfrak{S}_1} = \vdash_{\mathfrak{S}_2}$.

From $\mathfrak{S}_1 \subsetneq \mathfrak{S}_2$, we get that $\vdash_{\mathfrak{S}_1} \supseteq \vdash_{\mathfrak{S}_2}$.

On the other hand suppose $\Gamma \nvdash_{\mathfrak{S}_2} \varphi$. This implies in particular that $\varphi \neq r$. Assume $\varphi = p$, the case of $\varphi = q$ being the same. Then $p \notin \Gamma$. If $\Gamma = \emptyset, \{r\}$, clearly any $\mathfrak{s} \in \mathfrak{S}_1$ such that $\mathfrak{s}(p) = \mathsf{r}$ suffices. Now, assume $q \in \Gamma$, and notice that $\mathfrak{s} : x \mapsto \begin{cases} \mathsf{a} & \text{if } x = q, r \\ \mathsf{r} & \text{else} \end{cases} \in \mathfrak{S}_i(\Gamma) = \{\mathfrak{s} \in \mathfrak{S}_i \mid \mathfrak{s}(q) = \mathfrak{s}(r) = \mathsf{a}\}$, and in particular is in $\mathfrak{S}_1$, but it is not in $\mathfrak{S}_1(p)$, meaning that $\Gamma \nvdash_{\mathfrak{S}_1} \varphi$.

"Set (of things) -Set (of things)" consequence relation: a syntactic characterisation

The concept of symmetric consequence relation is first presented in Gerhard Gentzen's celebrated *"Untersuchungen über das logische Schliessen"* (1934) if one interpret his calculus of sequents as a metatheory for a "multiple-conclusion" logic (aka symmetric consequence relation on logical formulae).

Untersuchungen über das logische Schließen*). I.

Von

Gerhard Gentzen in Göttingen.

Übersicht.

Die folgenden Untersuchungen beziehen sich auf den Bereich der Prädikatenlogik [bei H.-A.[1]) „engerer Funktionenkalkül" genannt]. Diese umfaßt solche Schlüsse, die in allen Teilen der Mathematik immerzu gebraucht werden. Was noch zu ihnen hinzukommt, sind Axiome und Schlußweisen, die man den einzelnen Zweigen der Mathematik selbst zurechnen kann, z. B. in der elementaren Zahlentheorie die Axiome der natürlichen Zahlen, der Addition, Multiplikation und Potenzierung, sowie der Schluß der vollständigen Induktion; in der Geometrie die geometrischen Axiome.

3.1. Beweis des Hauptsatzes für LK-Herleitungen.

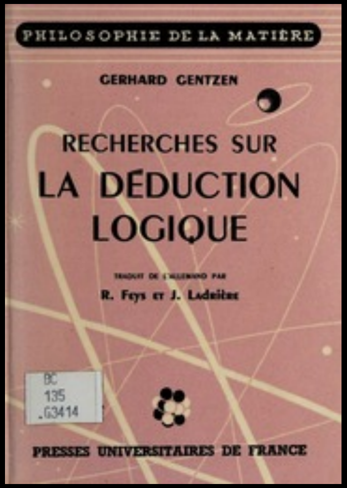
Wir führen (zur Erleichterung des Beweises) eine neue Schlußfigur ein, welche eine Abwandlung des Schnittes darstellt, wir nennen sie „Mischung".

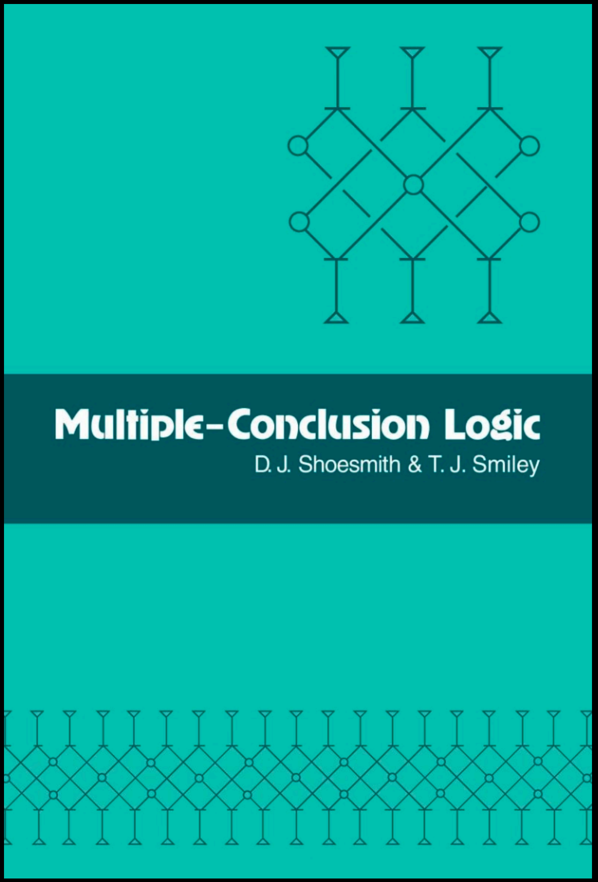
Das Schema hierfür lautet:

$$\frac{\Gamma \rightarrow \Theta \quad \Delta \rightarrow \Lambda}{\Gamma, \Delta^* \rightarrow \Theta^*, \Lambda}.$$

Dabei sind für Θ und Δ solche Reihen von Formeln, durch Kommata getrennt, einzusetzen, in welchen eine Formel der Gestalt $\mathfrak{M}$, wir nennen sie die „Mischformel", jeweils mindestens einmal (als Glied der Reihe) auftritt; für Θ^* und Δ^* sind dieselben Formelreihen einzusetzen, jedoch mit Weglassung sämtlicher (als Glieder der Reihe) vorkommenden Formeln der Gestalt $\mathfrak{M}$ ($\mathfrak{M}$ kann eine beliebige Formel sein). Für Γ und Λ sind, wie üblich, beliebige (eventuell leere) Reihen von Formeln, durch Kommata getrennt, einzusetzen.

"Set (of things) -Set (of things)" consequence relation: a syntactic characterisation





“Set (of things) -Set (of things)” consequence relation: a syntactic characterisation

Definition: Let $\mathscr{L}$ be a set. A relation $\vdash\ \subseteq \wp(\mathscr{L}) \times \wp(\mathscr{L})$ is called a **symmetric consequence relation**, if it satisfies the following for every $\Phi, \Theta, \Gamma, \Delta$:

- $\Gamma \vdash \Phi$, whenever $\Phi \cap \Gamma \neq \varnothing$ (Overlap)
- If $\Gamma \vdash \Phi$, then $\Gamma \cup \Delta \vdash \Phi \cup \Theta$ (Monotonicity / Dilution)
- If $\Gamma \cup \Theta_1 \vdash \Phi \cup \Theta_2$, for each quasi-partition (Θ_1, Θ_2) of Θ, then $\Gamma \vdash \Phi$ (Cut for sets)

Given a set Θ, a pair (Θ_1, Θ_2) is a quasi-partition of Θ whenever $\Theta = \Theta_1 \cup \Theta_2$. Hence in particular one of the member of the pair (Θ_1, Θ_2) can be empty. A quasi-partition is a partition when $\Theta_i \neq \varnothing$, for $i = 1,2$.

"Set (of things) -Set (of things)" consequence relation: a syntactic characterisation

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Fact: Assume $\vdash \subseteq \wp(\mathscr{L}) \times \wp(\mathscr{L})$ satisfies Overlap and Dilution. The following conditions are then equivalent:

- For each Θ, (Cut for sets) holds
- If $\Gamma \nvdash \Phi$ then there is a partition (Θ_T, Θ_F) of $\mathscr{L}$ such that $\Gamma \subseteq \Theta_T, \Phi \subseteq \Theta_F$ and $\Theta_T \nvdash \Theta_F$ (Cut for $\mathscr{L}$)
- If $\Gamma \cup \{\theta\} \vdash \Phi \cup \Theta$, for each $\theta \in \Theta$, and $\Gamma \cup \Theta \vdash \Phi$, then $\Gamma \vdash \Phi$ (Symmetric Cut)

"Set (of things) -Set (of things)" consequence relation: a syntactic characterisation

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- If $\Gamma \vdash \Phi$, then $\Gamma \cup \Delta \vdash \Phi \cup \Theta$ (Monotonicity / Dilution)
- If $\Gamma \cup \Theta_1 \vdash \Phi \cup \Theta_2$, for each quasi-partition (Θ_1, Θ_2) of Θ, then $\Gamma \vdash \Phi$ (Cut for sets)

- A symmetric consequence relation is called **finitary** if, for every Γ, Φ:
  - If $\Gamma \vdash \Phi$, then there are finite $\Delta \subseteq \Gamma$ and $\Theta \subseteq \Phi$ such that $\Delta \vdash \Theta$ (Finitariness)

One may "read" a symmetric consequence relation as something like $\bigwedge \Gamma \vdash \bigvee \Phi$

"Set (of things) -Set (of things)" consequence relation: a semantic characterisation

"Set (of things) -Set (of things)" consequence relation: a semantic characterisation

Consider a valuation $\mathfrak{s} : \mathscr{L} \to \{\mathsf{a}, \mathsf{r}\}$. We extend $\mathfrak{s} : \wp(\mathscr{L}) \to \{\mathsf{a}, \mathsf{r}, \star\}$ by setting

$$\mathfrak{s} : \Gamma \mapsto \begin{cases} \mathsf{a} & \text{if } \forall \gamma \in \Gamma : \mathfrak{s}(\gamma) = \mathsf{a} \\ \mathsf{r} & \text{if } \forall \gamma \in \Gamma : \mathfrak{s}(\gamma) = \mathsf{r} \\ \star & \text{else} \end{cases}.$$

Given a collection $\mathfrak{S}$ and a set of "things" $\Gamma \subseteq \mathscr{L}$, we then define $\mathfrak{S}_\mathsf{a}(\Gamma) = \{\mathfrak{s} \in \mathfrak{S} \mid \mathfrak{s}(\Gamma) = \mathsf{a}\}$ and $\mathfrak{S}_\mathsf{r}(\Gamma) = \{\mathfrak{s} \in \mathfrak{S} \mid \mathfrak{s}(\Gamma) = \mathsf{r}\}$,

- *Definition*: Let $\mathscr{L}$ be a set and, and $\mathfrak{S}$ a collection of possible worlds. Then, for every $\Gamma, \Phi \subseteq \mathscr{L}$, we define

$$\Gamma \vdash_{\mathfrak{S}} \Phi \text{ iff } \mathfrak{S}_\mathsf{a}(\Gamma) \cap \mathfrak{S}_\mathsf{r}(\Phi) = \emptyset.$$

Equivalently, $\Gamma \vdash_{\mathfrak{S}} \Phi$ iff there is no $\mathfrak{s} \in \mathfrak{S}$ such that $\mathfrak{s}(\Gamma) = \mathsf{a}$ and $\mathfrak{s}(\Phi) = \mathsf{a}$.

"Set (of things) -Set (of things)" consequence relation: a semantic characterisation

Consider a valuation $\mathfrak{s} : \mathscr{L} \to \{\mathsf{a}, \mathsf{r}\}$. We extend $\mathfrak{s} : \wp(\mathscr{L}) \to \{\mathsf{a}, \mathsf{r}, \star\}$ by setting

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- *Definition*: Let $\mathscr{L}$ be a set and, and $\mathfrak{S}$ a collection of possible worlds. Then, for every $\Gamma, \Phi \subseteq \mathscr{L}$, we define

$$\Gamma \vdash_{\mathfrak{S}} \Phi \text{ iff } \mathfrak{S}_{\mathsf{a}}(\Gamma) \cap \mathfrak{S}_{\mathsf{r}}(\Phi) = \varnothing.$$

- *Fact*: Let $\mathscr{L}$ be a set and, and $\mathfrak{S}$ a collection of possible worlds. Then the relation $\vdash_{\mathfrak{S}}$ is a symmetric consequence relation.

Proof: For overlap, assume $\Phi \cap \Gamma \neq \varnothing$. This means that if $\mathfrak{s}(\Gamma) = \mathsf{a}$, then $\mathfrak{s} \notin \mathfrak{S}_{\mathsf{r}}(\Phi)$, hence $\Gamma \vdash_{\mathfrak{S}} \Phi$. Monotonicity being obvious, one can also check that cut for sets holds (EXERCISE).

"Set (of things) -Set (of things)" consequence relation: completeness

- *Theorem (completeness for symmetric consequence)*: Let $(\mathscr{L}, \vdash)$ be a symmetric consequence system. Then it is always possible to find a class of possible words $\mathfrak{S}$ over A such that $\vdash \, = \, \vdash_{\mathfrak{S}}$.

Proof: Fix an arbitrary pair (Γ, Φ) such that $\Gamma \nvdash \Phi$. Then, by Cut for $\mathscr{L}$, there is a partition (Θ_a, Θ_r) of $\mathscr{L}$ such that $\Gamma \subseteq \Theta_a, \Phi \subseteq \Theta_r$ and $\Theta_a \nvdash \Theta_r$. So, let $\mathfrak{S}^\star$, the collection of the valuations (characteristic functions) corresponding to such pairs so that assign **a** to "things" in Θ_a and **r** in Θ_r. Consider the induced consequence relation $\vdash_{\mathfrak{S}^\star}$.

Again, the idea is that the partition (Θ_a, Θ_r) of $\mathscr{L}$ induces a possible world via the corresponding accept/truth set (predicate) Θ_a, and its dual (complementary) reject/false set (predicate) Θ_r.

"Set (of things) -Set (of things)" consequence relation: completeness

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  For the other direction, let us assume that, for some pair (Ξ, Λ), $\Xi \vdash \Lambda$ holds. Towards a contradiction, let us suppose $\Xi \nvdash_{\mathfrak{S}^\star} \Lambda$. This means that there is a pair (Γ, Φ) and a partition (Θ_a, Θ_r) of $\mathscr{L}$ with $\Gamma \subseteq \Theta_a, \Phi \subseteq \Theta_r$ such that $\Gamma \nvdash \Phi$, $\Theta_a \nvdash \Theta_r$ and $\Xi \subseteq \Theta_a, \Lambda \subseteq \Theta_r$. However, $\Theta_a \vdash \Theta_r$ by applying dilution to $\Xi \vdash \Lambda$, a contradiction. Hence $\Xi \vdash_{\mathfrak{S}^\star} \Lambda$.

"Set (of things) -Set (of things)" consequence relation: absoluteness

- *Theorem (absoluteness for symmetric consequence)*: Any semantics characterises a unique symmetric consequence system.

  Proof: Fix an arbitrary family of valuations $\mathfrak{S}$ and its induced symmetric consequence system $(\mathscr{L}, \vdash_{\mathfrak{S}})$. Consider the class $\mathfrak{S}(\vdash_{\mathfrak{S}}) := \{\mathfrak{s} \subseteq \mathscr{L} \mid \forall \Gamma, \Phi : (\Gamma \vdash_{\mathfrak{S}} \Phi \wedge \Gamma \subseteq \mathfrak{s}) \rightarrow (\Phi \cap \mathfrak{s} \neq \emptyset)\}$ of valuations that respect $\vdash_{\mathfrak{S}}$. We need to check that $\mathfrak{S}(\vdash_{\mathfrak{S}}) = \mathfrak{S}$.

"Set (of things) -Set (of things)" consequence relation: absoluteness

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  For the other direction, suppose $\mathfrak{s} \notin \mathfrak{S}$, and consider the partition $(\mathfrak{s}, \mathscr{L} \backslash \mathfrak{s})$. Recall that $\Gamma \vdash_{\mathfrak{S}} \Phi$ iff there is no $\mathfrak{s}' \in \mathfrak{S}$ such that $\Gamma \subseteq \mathfrak{s}'$ and $\Phi \subseteq \mathscr{L} \backslash \mathfrak{s}'$. Since trivially there is no $\mathfrak{s}' \in \mathfrak{S}$ such that $\mathfrak{s} \subseteq \mathfrak{s}'$ and $\mathscr{L} \backslash \mathfrak{s} \subseteq \mathscr{L} \backslash \mathfrak{s}'$, we have that $\mathfrak{s} \vdash_{\mathfrak{S}} \mathscr{L} \backslash \mathfrak{s}$. But again, notice that trivially despite $\mathfrak{s} \vdash_{\mathfrak{S}} \mathscr{L} \backslash \mathfrak{s}$ and $\mathfrak{s} \subseteq \mathfrak{s}$, $\mathscr{L} \backslash \mathfrak{s} \cap \mathfrak{s} = \emptyset$, meaning that $\mathfrak{s} \notin \mathfrak{S}(\vdash_{\mathfrak{S}})$.

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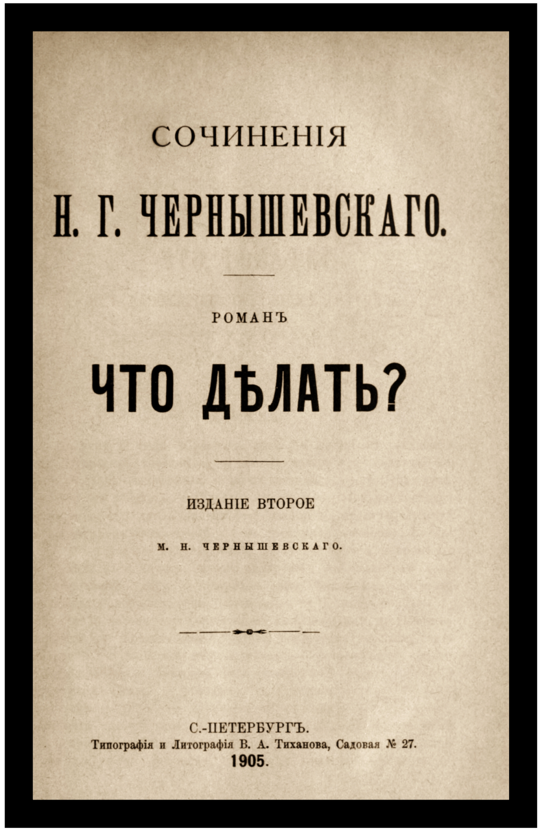
Varieties of desirabilities

Theory of rationality


СОЧИНЕНІЯ

Н. Г. ЧЕРНЫШЕВСКАГО.

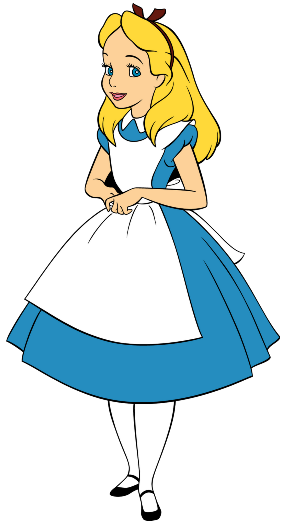
РОМАНЪ

ЧТО ДѢЛАТЬ?

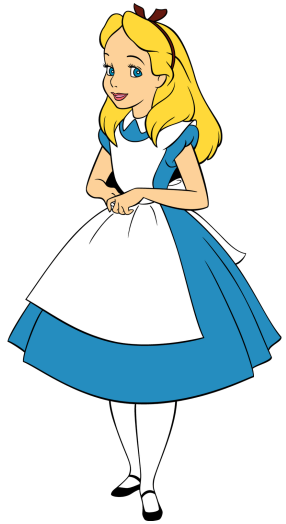
ИЗДАНІЕ ВТОРОЕ

М. Н. ЧЕРНЫШЕВСКАГО.

С.-ПЕТЕРБУРГЪ.
Типографія и Литографія В. А. Тиханова, Садовая № 27.
1905.

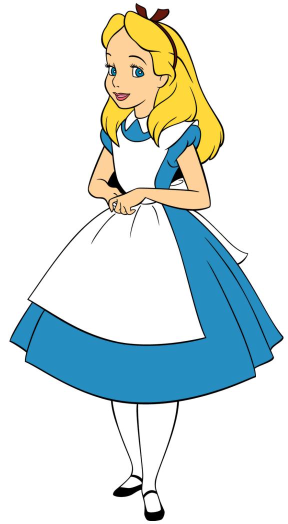


Theory(-ies) of desirability as theory(-ies) of rational decision making under uncertainty

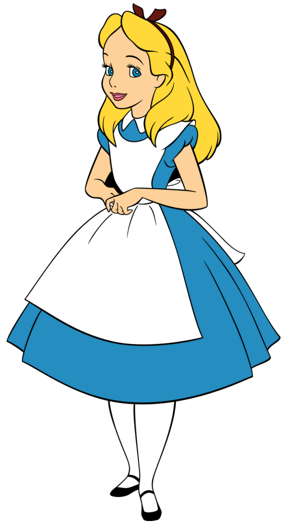


Theory of rationality

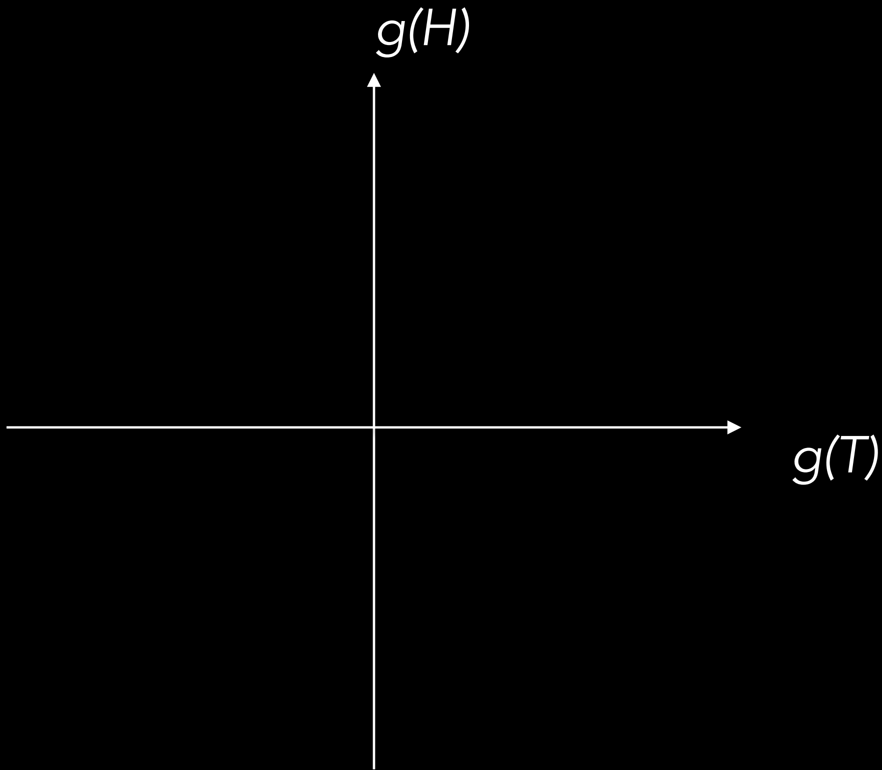
This is for simplicity. When needed, we make it explicit though.

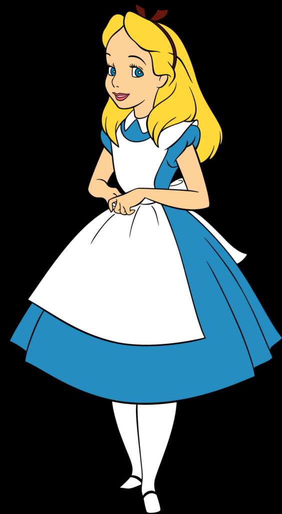


So, from now on here for simplicity we are essentially over $\mathbb{R}^n$. But make explicit when needed.

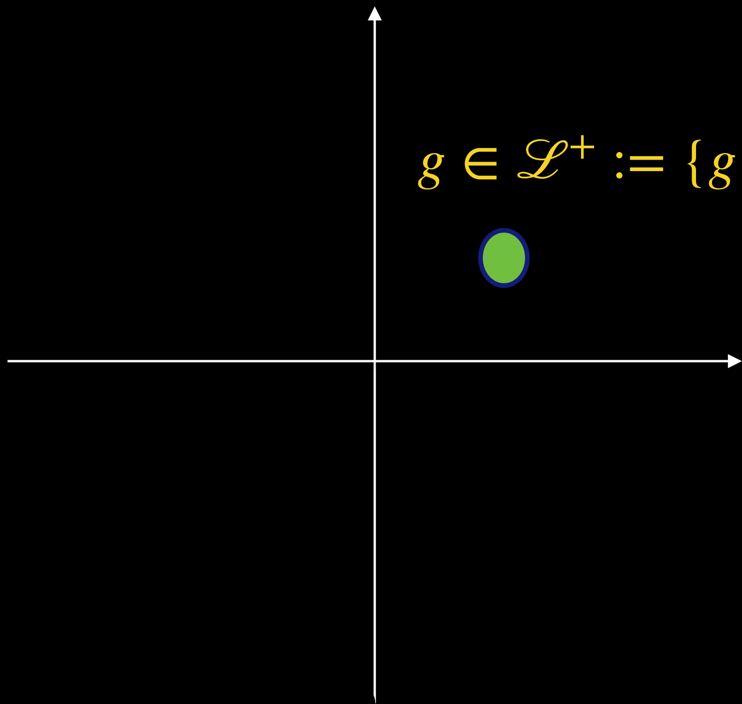


The coin experiment

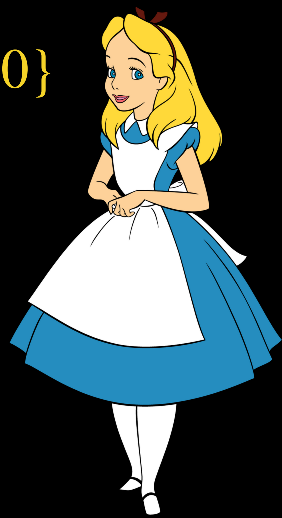




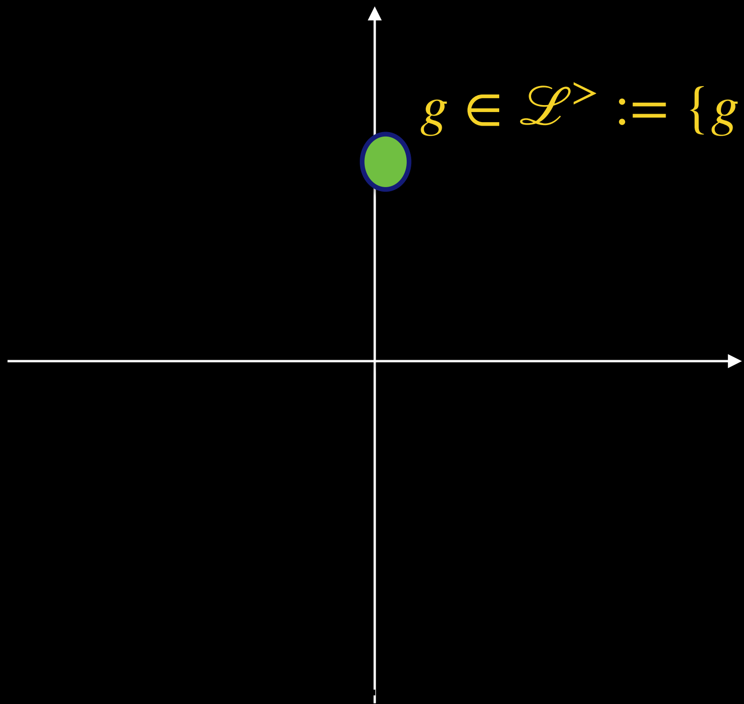
The coin experiment: accepting sure gain



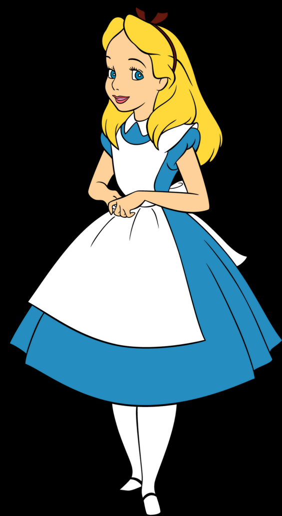
$$g \in \mathscr{L}^{+} := \{g \mid \inf g > 0\}$$



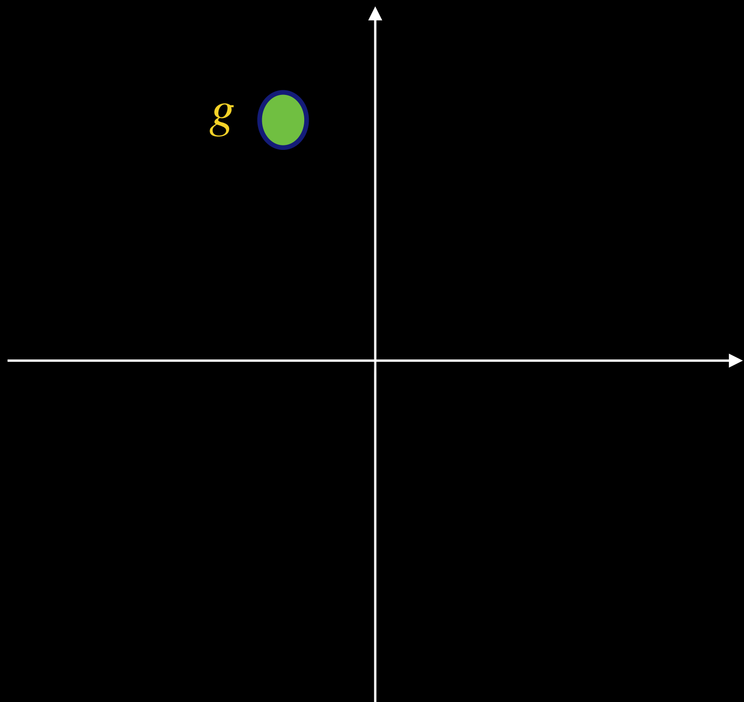
The coin experiment: accepting partial gain

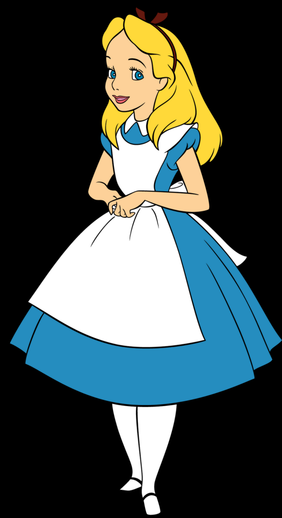


$$g \in \mathscr{L}^{>} := \{g \mid g > 0\}$$



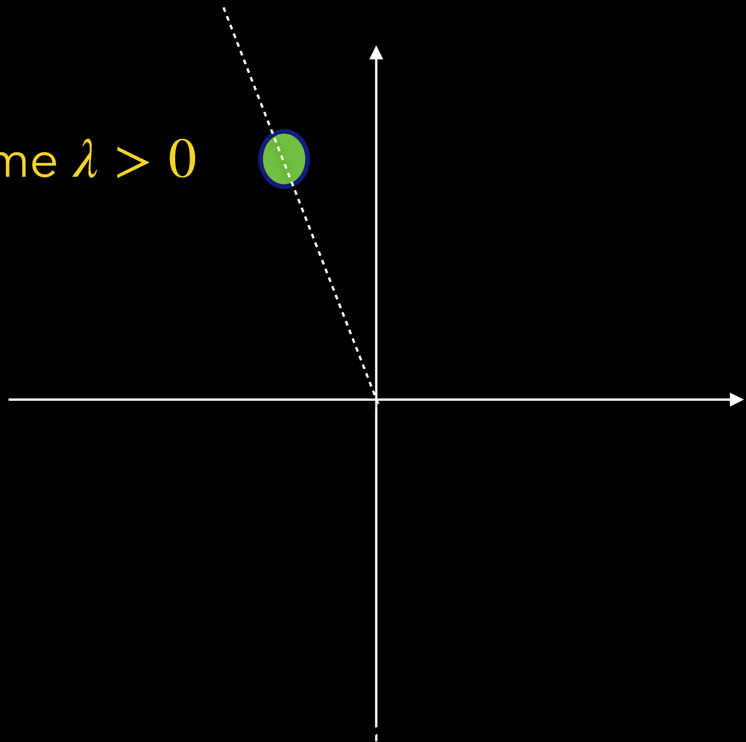
The coin experiment

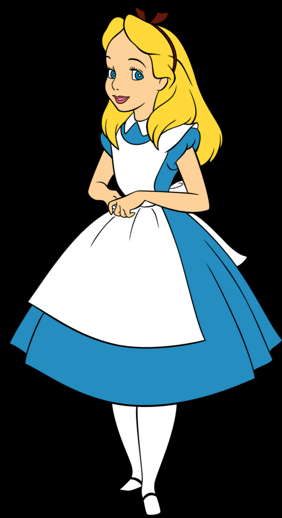




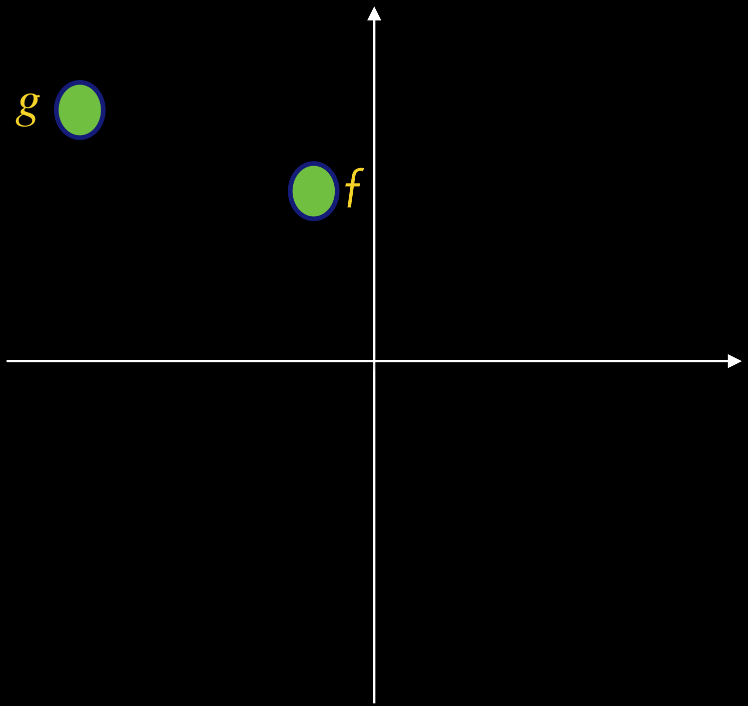
The coin experiment: positive scaling

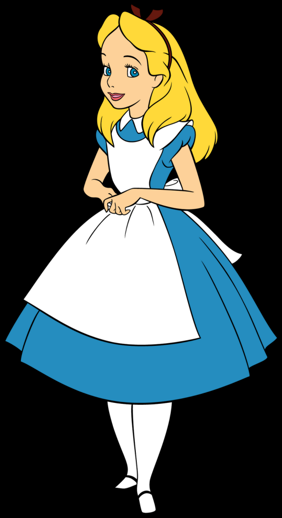
$f = \lambda g,$ for some $\lambda > 0$



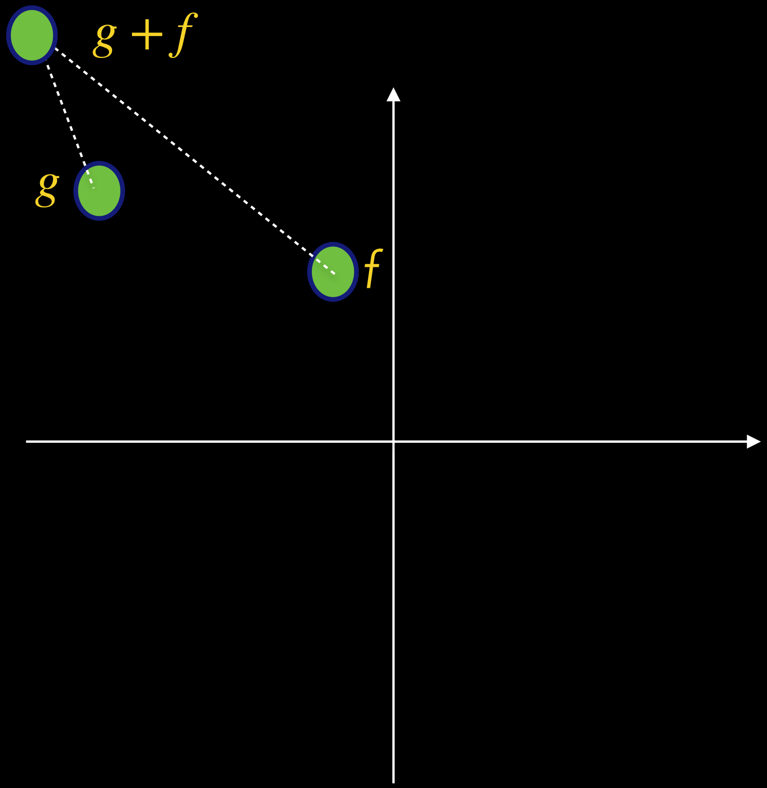


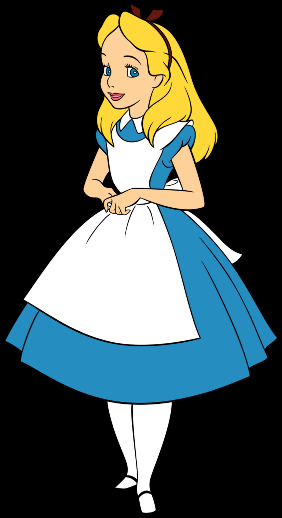
The coin experiment



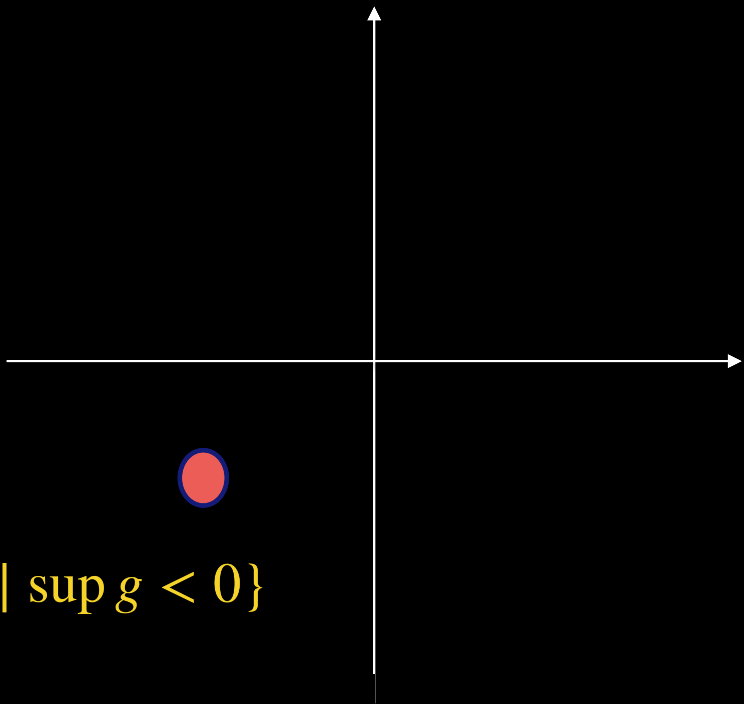


The coin experiment: addition

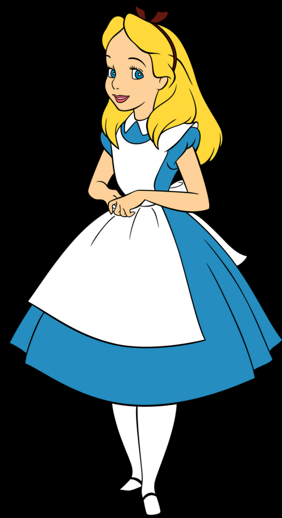




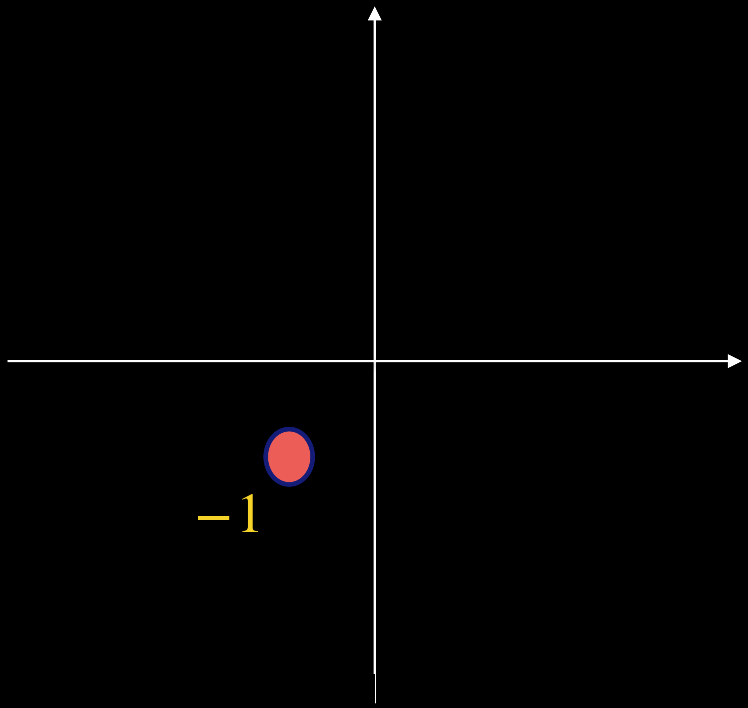
The coin experiment: avoiding sure loss

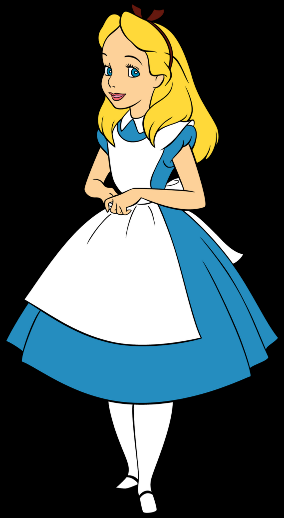


$$g \in \mathscr{L}^{-} := \{g \mid \sup g < 0\}$$



The coin experiment: avoiding the negative unit





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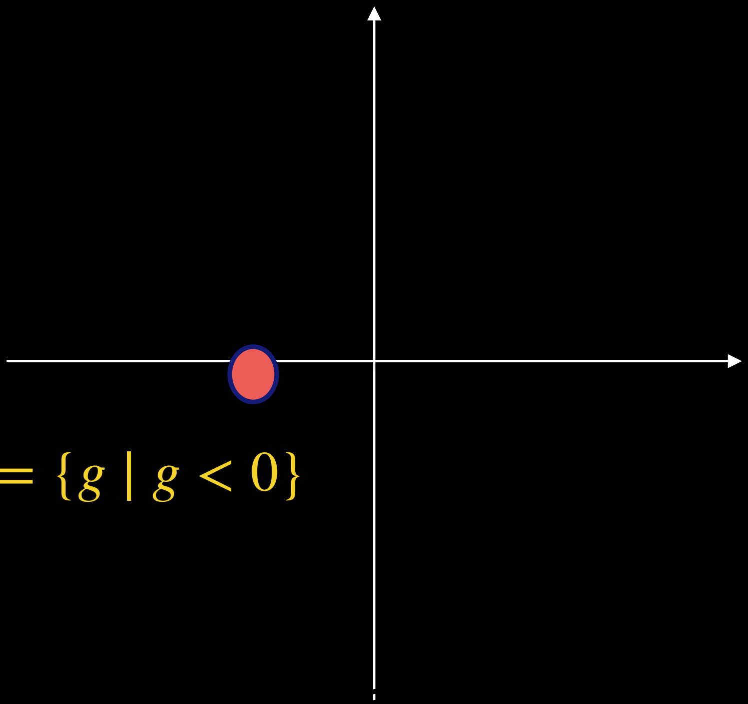
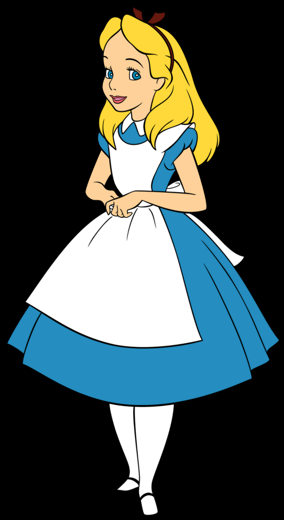
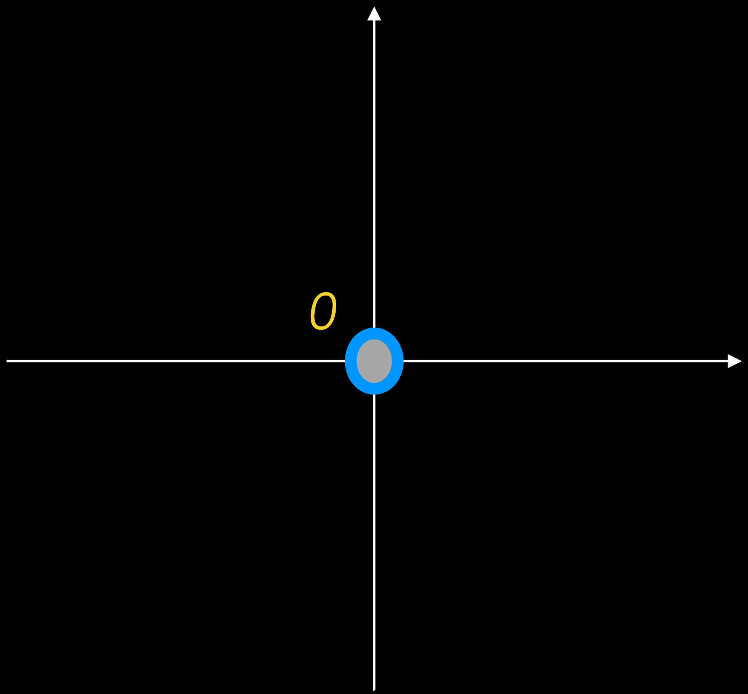
The coin experiment: avoiding partial loss

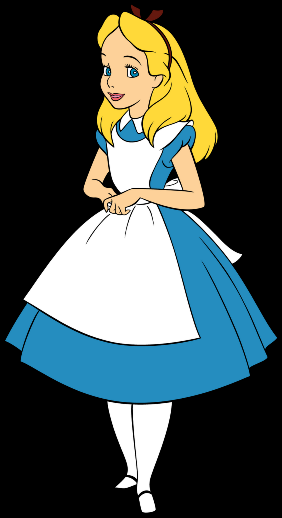


$$g \in \mathscr{L}^{<} := \{g \mid g < 0\}$$

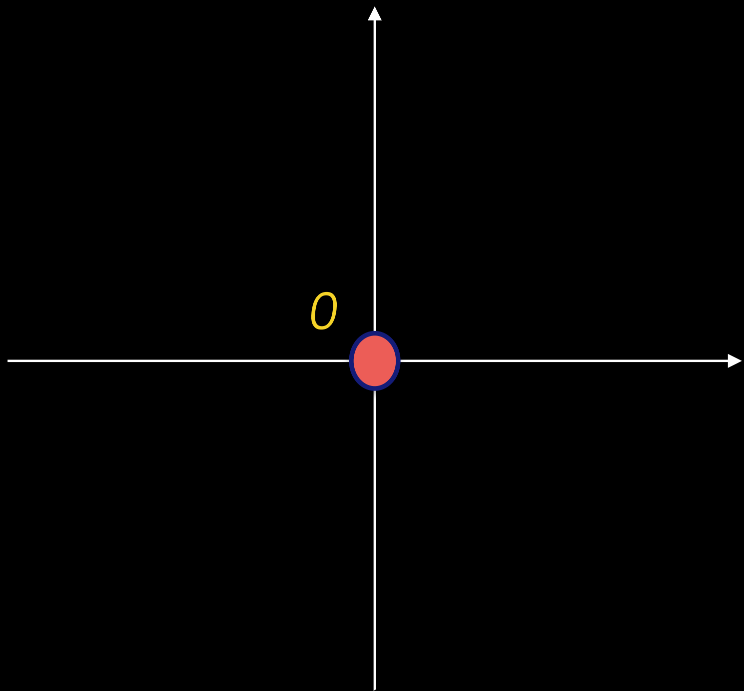


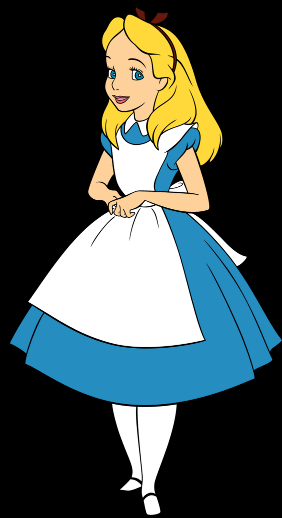
The coin experiment: the status quo





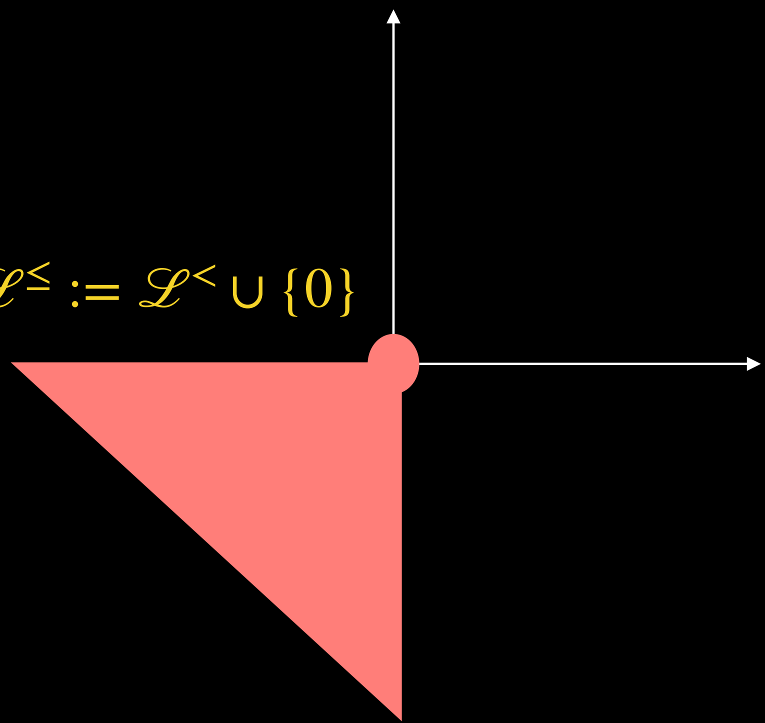
The coin experiment: avoiding status quo

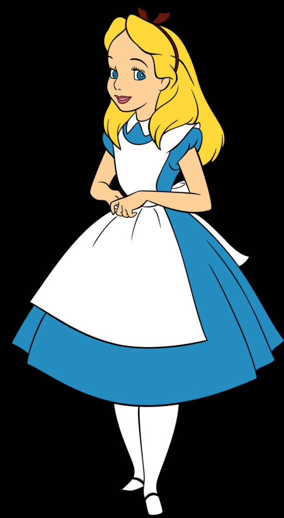




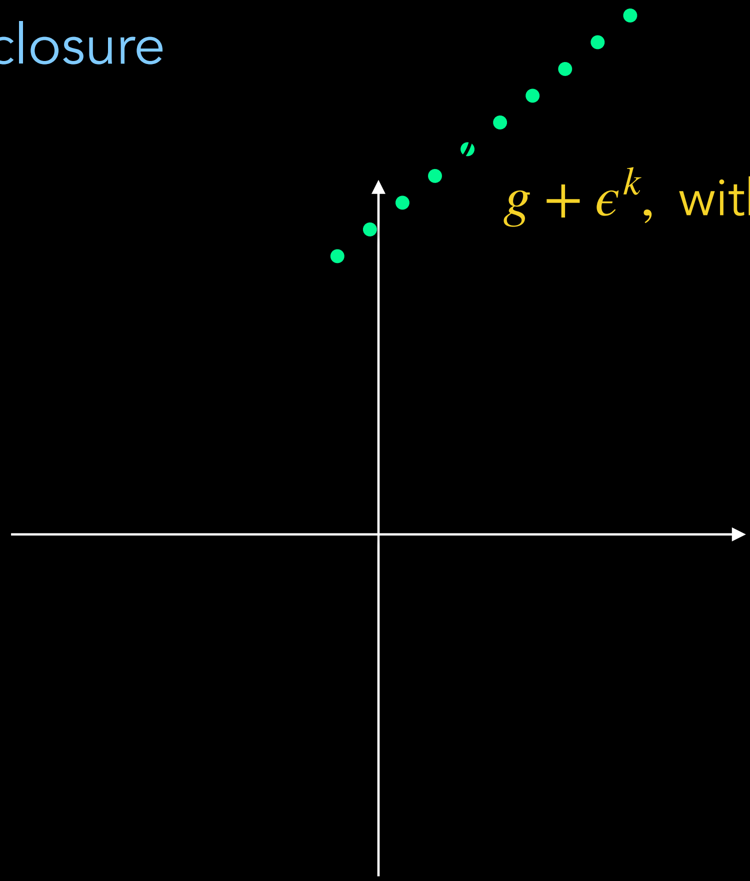
The coin experiment: avoiding non-positivity

$$\mathscr{L}^{\leq} := \mathscr{L}^{<} \cup \{0\}$$

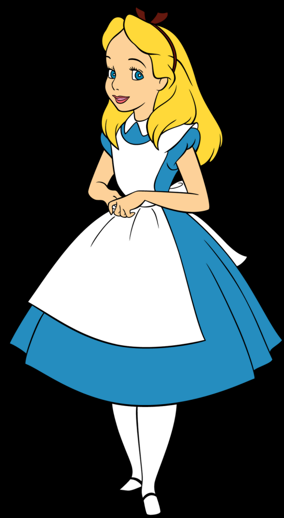




The coin experiment: closure



$g + \epsilon^k$, with $\epsilon > 0$



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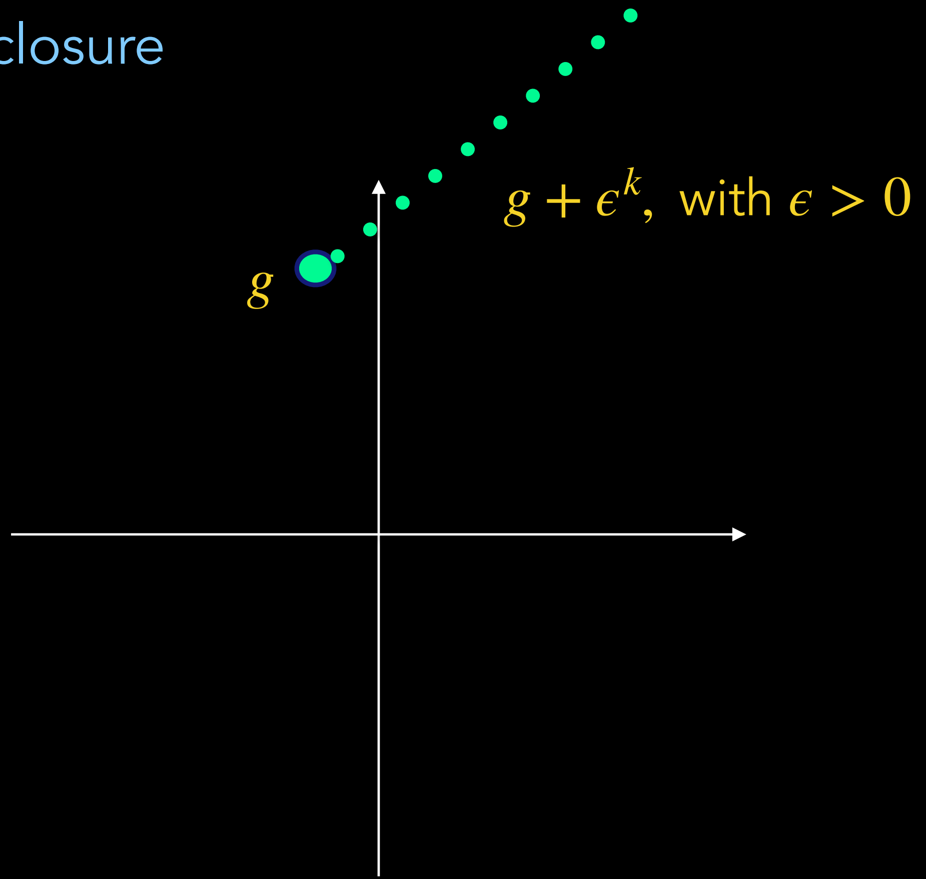
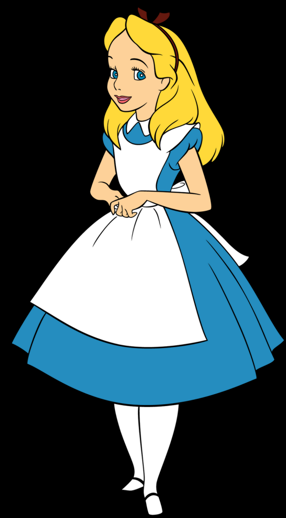


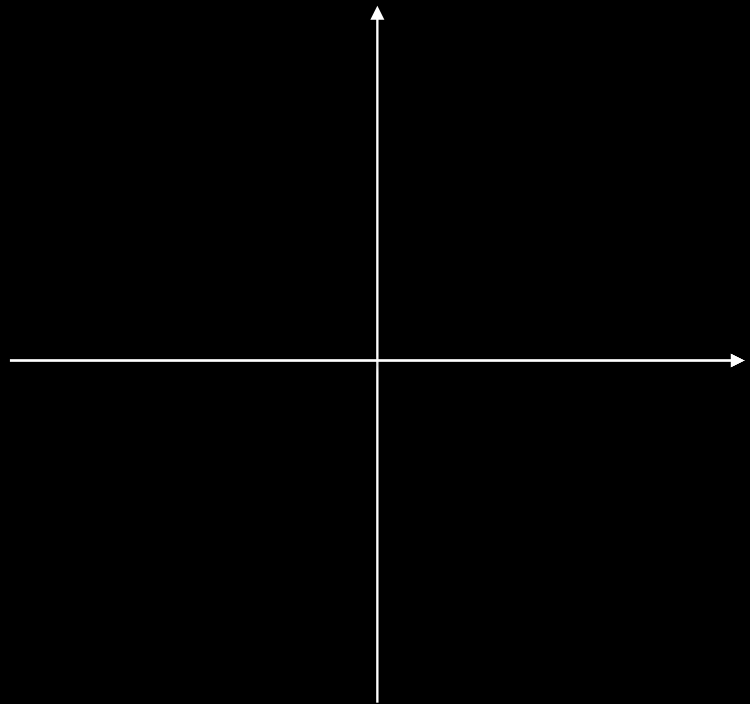


Conditions on a set $\mathscr{K}$ of gambles

(ASG) accepting sure gain	$\mathscr{L}^+ \subseteq \mathscr{K}$
(APG) accepting partial gain	$\mathscr{L}^> \subseteq \mathscr{K}$
(PS) positive scaling	$\lambda \mathscr{K} \subseteq \mathscr{K}, \lambda > 0$
(ADD) addition	$\mathscr{K} + \mathscr{K} \subseteq \mathscr{K}$
(ASL) avoiding sure loss	$\mathscr{L}^- \cap \mathscr{K} = \varnothing$
(APL) avoiding partial loss	$\mathscr{L}^< \cap \mathscr{K} = \varnothing$
(ANP) avoiding non positivity	$\mathscr{L}^\leq \cap \mathscr{K} = \varnothing$
(ASQ) avoiding status quo	$0 \notin \mathscr{K}$
(ANU) avoiding negative unit	$-1 \notin \mathscr{K}$
(CL) closure	$f \in \mathscr{K}$, if $\exists \epsilon \in (0,1) \forall k > 0 : f + \epsilon^k \in \mathscr{K}$

Theory of desirable gambles: coherence

- *Definition*: A set $\mathscr{K} \subseteq \mathscr{L}(\Omega)$ is **coherent** if it satisfies
  - (APG) $\mathscr{L}^{>} \subseteq \mathscr{K}$
  - (PS) $\lambda\mathscr{K} \subseteq \mathscr{K}$, for $\lambda > 0$
  - (ADD) $\mathscr{K} + \mathscr{K} \subseteq \mathscr{K}$
  - (APL) $\mathscr{L}^{<} \cap \mathscr{K} = \emptyset$



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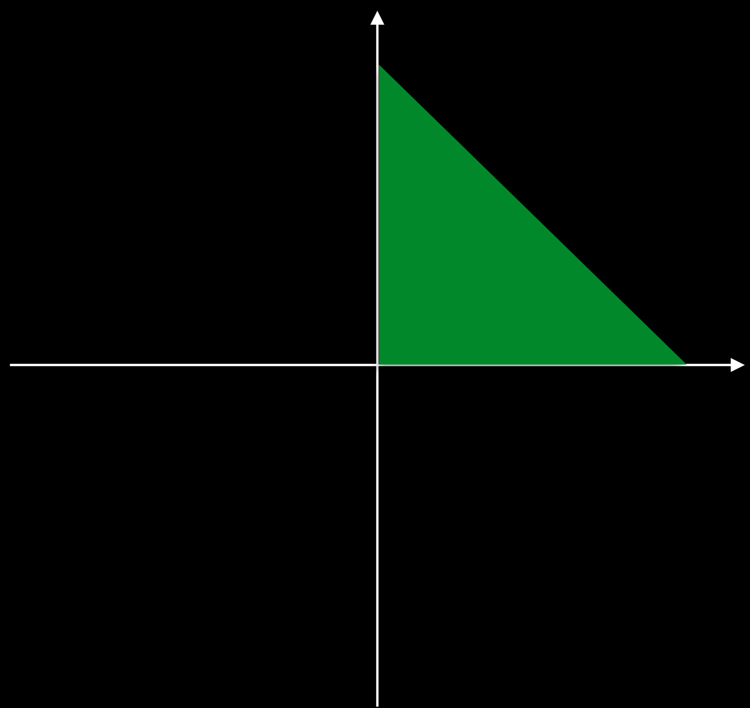

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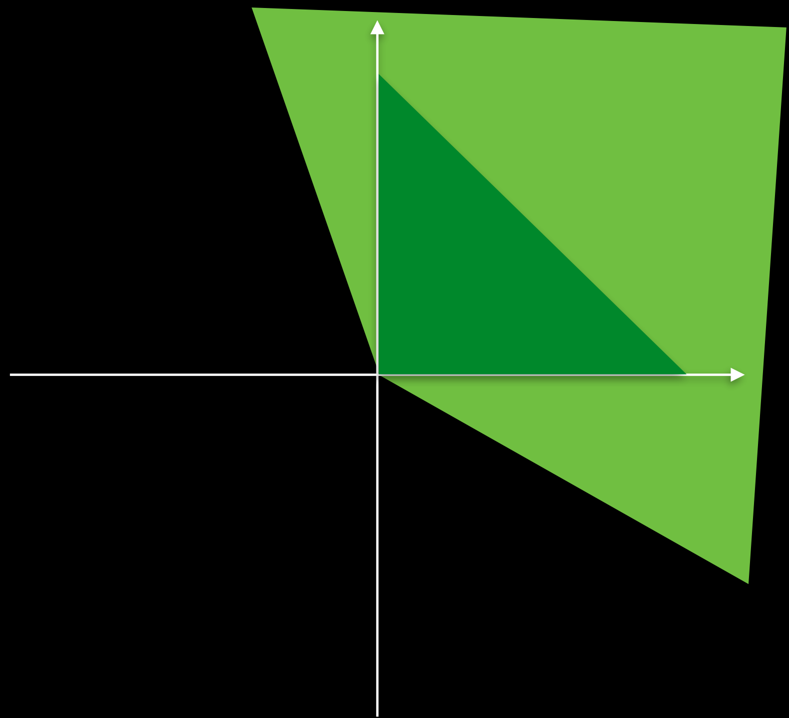

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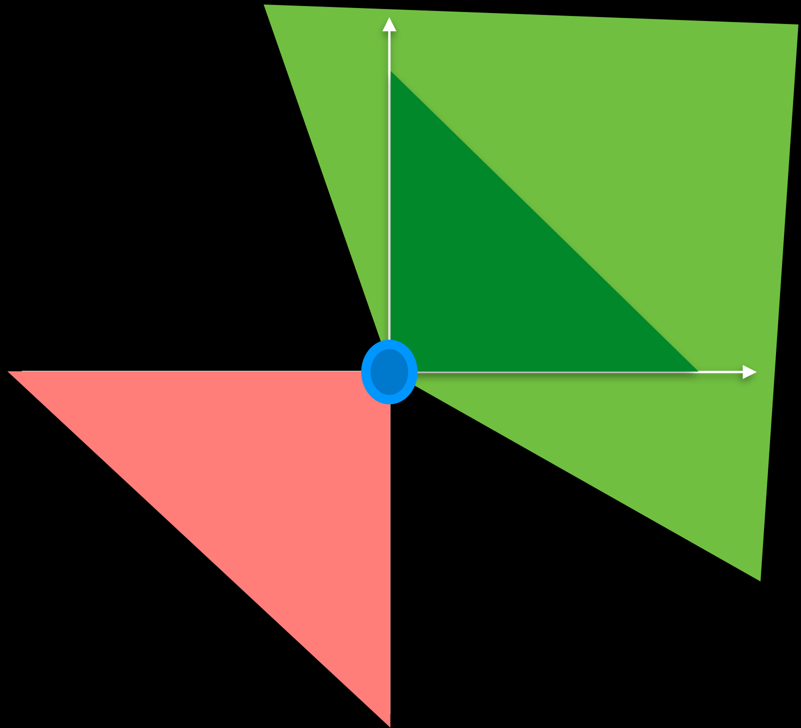

Theory of desirable gambles: coherence (a bit stronger)

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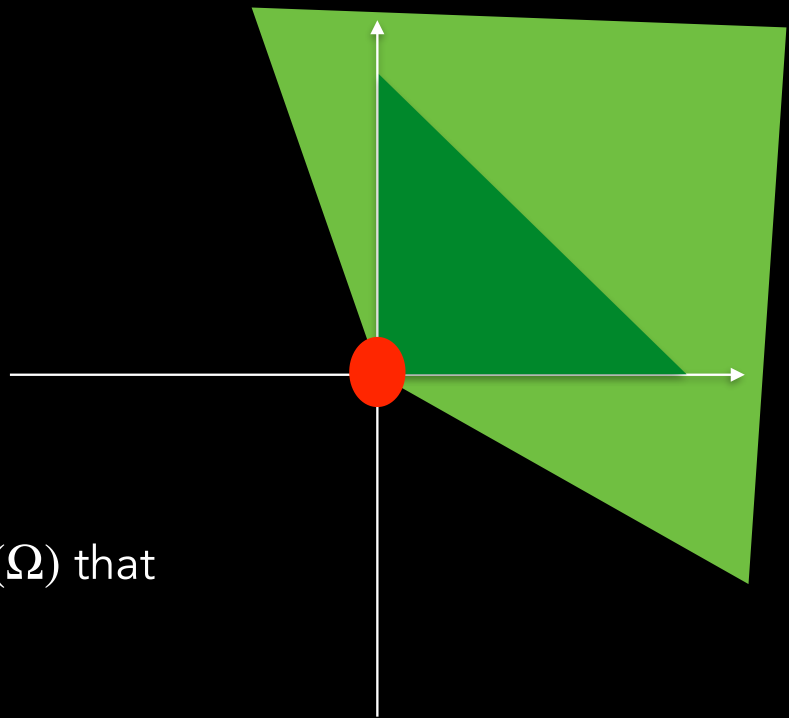
  (ASQ) $0 \notin \mathscr{K}$

The collection of all set of desirable gambles over $\mathscr{L}(\Omega)$ that are coherent is then denoted by $\mathbf{C}(\mathscr{L}(\Omega))$



Theory of desirable gambles: coherence (a bit stronger)

Fact: (APG+PS+ADD+ASQ) implies (APL) and thus (ANP)

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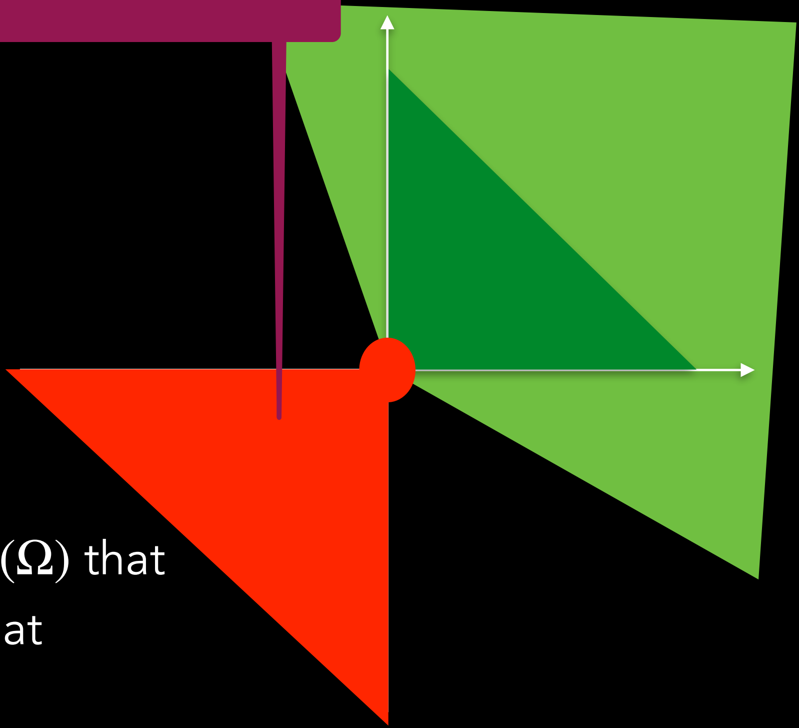

The collection of all set of desirable gambles over $\mathscr{L}(\Omega)$ that are coherent is then denoted by $\mathbf{C}(\mathscr{L}(\Omega))$. We say that $\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega))$ is maximally coherent if there is no $\mathscr{K}' \in \mathbf{C}(\mathscr{L}(\Omega))$ such that $\mathscr{K} \subsetneq \mathscr{K}'$, and denote the collection of maximally coherent sets by $\mathbf{M}(\mathscr{L}(\Omega))$.

Theory of desirable gambles: the structure of coherent sets

- *Lemma (Couso & Moral 2011)*: Let $\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega))$ but $g \notin \mathscr{K}$, with $g \neq 0$. Then the following hold
  1. posi$(\mathscr{A} \cup \{-g\}) \in \mathbf{C}(\mathscr{L}(\Omega))$
  2. if $-g \notin \mathscr{K}$, then $\mathscr{K}^g :=$ posi$(\mathscr{A} \cup \{g\} \cup \{-g + f \mid f \in \mathscr{K}\}) \in \mathbf{C}(\mathscr{L}(\Omega))$

  Where the positive hull operator is defined as posi$(X) := \{\mu f + \lambda g \mid f, g \in X,$ and $\mu, \lambda > 0\}$.

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  Where the positive hull operator is defined as posi$(X) := \{\mu f + \lambda g \mid f, g \in X,$ and $\mu, \lambda > 0\}$.

Since one can check that the first point of the Lemma readily implies that any $\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega))$ such that either $g \in \mathscr{K}$ or $-g \in \mathscr{K}$, for each $g \in \mathscr{L}(\Omega)\backslash\{0\}$, cannot be extended to a coherent strict supset, we get that :

- *Theorem (Couso & Moral 2011)*: The maximal coherent sets of desirable gambles over $\mathscr{L}(\Omega)$ are the semispaces at the origin (i.e. convex sets $\mathscr{K} \subseteq \mathscr{L}(\Omega)$ without the origin 0 and such that either $g \in \mathscr{K}$ or $-g \in \mathscr{K}$ for each $g \in \mathscr{L}(\Omega)\backslash\{0\}$) that contain the positive orthant $\mathscr{L}(\Omega)^{>}$.

Theory of desirable gambles: the structure of coherent sets

The construction is reminiscent of the ultrafilter procedure used e.g. to prove compactness in classical logic

- *Lemma* (C[...] hold
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  2. if $-g \notin \mathscr{K}$, then $\mathscr{K}^g :=$ posi$(\mathscr{A} \cup \{g\} \cup \{-g + f \mid f \in \mathscr{K}\}) \in \mathbf{C}(\mathscr{L}(\Omega))$

  Where the positive hull operator is defined as posi$(X) := \{\mu f + \lambda g \mid f, g \in X,$ and $\mu, \lambda > 0\}$.

The second point of the Lemma, joint with the previous characterisation of maximal coherent sets, provides a recursive way to complete a coherent set $\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega))$ and construct a maximal one containing it by including a gamble and excluding its "inverse/negation":

- Set $\mathscr{K}_0 := \mathscr{K}$.
- Consider $\mathscr{K}_n \supseteq \mathscr{K}$, if $\mathscr{K}_n \in \mathbf{M}(\mathscr{L}(\Omega))$, stops.
- Else for some $g \in \mathscr{L}(\Omega) \backslash \{0\}$ both $g, -g \notin \mathscr{K}_n$, and define $\mathscr{K}_{n+1} := (\mathscr{K}_n)^g$.

One can actually prove that there is a finite k such that $\mathscr{K}_{k+1} = \mathscr{K}_k \in \mathbf{M}(\mathscr{L}(\Omega))$. That is

Theorem (Couso & Moral 2011): If $\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega))$, there is $\mathscr{K}' \in \mathbf{M}(\mathscr{L}(\Omega))$ such that $\mathscr{K}' \supseteq \mathscr{K}$.

Theory of desirable gambles: the structure of coherent sets

We thus have that

- *Theorem (Couso & Moral 2011)*: If $\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega))$, then $\mathscr{K} = \bigcap \{\mathscr{K}' \in \mathbf{M}(\mathscr{L}(\Omega)) \mid \mathscr{K} \subseteq \mathscr{K}'\}$.

  Proof: Obviously $\mathscr{K} \subseteq \bigcap \{\mathscr{K}' \in \mathbf{M}(\mathscr{L}(\Omega)) \mid \mathscr{K} \subseteq \mathscr{K}'\}$, and assume there is $g \in \left(\bigcap \{\mathscr{K}' \in \mathbf{M}(\mathscr{L}(\Omega)) \mid \mathscr{K} \subseteq \mathscr{K}'\}\right) \backslash \mathscr{K}$. Then posi($\mathscr{K} \cup \{-g\}) \in \mathbf{C}(\mathscr{L}(\Omega))$ and there is $\mathscr{K}' \in \mathbf{M}(\mathscr{L}(\Omega))$ such that $\mathscr{K}' \supseteq \mathscr{K}$. But by construction $g, -g' \in \mathscr{K}'$, a contradiction.

And thus we get that:

Corollary: $(\mathbf{C}(\mathscr{L}(\Omega)) \cup \{\mathscr{L}\}, \subseteq)$ is a dually atomic complete lattice.

Theory of desirable gambles: natural extension, first try

Sets of desirable gambles: Conditioning, representation, and precise probabilities (Couso & Moral 2011)

Let $\Omega = \{\omega_1, \dots, \omega_n\}$ denote the (finite) set of outcomes. We assume that there is an unknown outcome value belonging to Ω. A *gamble* on Ω is a bounded mapping from Ω to $\mathbb{R}$, i.e., $X : \Omega \to \mathbb{R}$. Gambles are used to represent an agent's beliefs and information. If an agent accepts a gamble X, then the value $X(\omega)$ represents the reward she would obtain if ω is the true unknown outcome (this value can be negative, in which case it represents a loss).

Let $\mathcal{L}$ denote the set of all gambles defined on Ω. For $X, Y \in \mathcal{L}$, let $X \geq Y$ mean that $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$, and let $X > Y$ mean that $X \geq Y$ and $X(\omega) > Y(\omega)$ for some $\omega \in \Omega$.

A subset $\mathcal{D}$ of $\mathcal{L}$ is said to be a *coherent set of desirable gambles* relative to $\mathcal{L}$ [14] when it satisfies the following four axioms:

D1. $0 \notin \mathcal{D}$,
D2. if $X \in \mathcal{L}$ and $X > 0$ then $X \in \mathcal{D}$,
D3. if $X \in \mathcal{D}$ and $c \in \mathbb{R}$ with $c > 0$ then $cX \in \mathcal{D}$,
D4. if $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ then $X + Y \in \mathcal{D}$.

If $\mathcal{F}$ is an arbitrary set of gambles, then the set of all gambles obtained by applying axioms D2, D3, and D4 is called the *set of gambles generated* by $\mathcal{F}$ and it is denoted by $\overline{\mathcal{F}}$. If this set is coherent ($0 \notin \overline{\mathcal{F}}$) then it will be called its *natural extension* (the minimum coherent set containing $\mathcal{F}$). If $0 \in \overline{\mathcal{F}}$ we will say that $\mathcal{F}$ does not avoid null gain. Natural extension will make sense only when $\mathcal{F}$ avoids null gain. If $X < 0$ and $X \in \overline{\mathcal{F}}$, we will say that $\mathcal{F}$ does not *avoid partial loss*.

It is immediate that

$$\overline{\mathcal{F}} = \left\{ \sum_{i=1}^{n} \lambda_i X_i \ : \ \lambda_i > 0, \ [X_i \in \mathcal{F} \text{ or } X_i > 0], \ n \geq 1 \right\} \tag{2}$$

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Theory of desirable gambles: natural extension, first try

gambles: Conditioning,
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Fact: The operator **posi**$(\,\cdot \cup \mathscr{L}^{>}) : \wp(\mathscr{L}) \to \wp(\mathscr{L})$ is a closure operator, and the collection of its closed sets is a strict supset of $\mathbf{C}(\mathscr{L}) \cup \{\mathscr{L}\}$, since e.g. $\mathscr{L}^{>} \cup \{0\} \neq \mathscr{L}$ is closed but not coherent, or any closed halfspace containing the positive orthant and the origin in the boundary.

Theory o

Here APL and not ASQ is the defining coherence condition (hence weaker form of coherence). Still, we have that the operator **posi**$(\cdot \cup \mathscr{L}^{>}) : \wp(\mathscr{L}) \to \wp(\mathscr{L})$ is a closure operator, but the collection of its closed sets is a strict supset of $\mathbf{C}(\mathscr{L}) \cup \{\mathscr{L}\}$, since, again, it includes any closed halfspace containing the positive orthant and the origin in the boundary.

Ch.1 Desirability (Erik Quaeghebeur 2014)

1.2.4 Natural extension

Given an assessment $\mathcal{A}$, the fact that all gambles in $\mathcal{L}^+(\mathcal{X})$ must be desirable, and the constructive rationality criteria, there is a *natural extension*, defined as

$$\mathcal{E}(\mathcal{A}) := \text{posi}\,(\mathcal{A} \cup \mathcal{L}^+(\mathcal{X})) = \text{posi}\,\mathcal{A} \cup \mathcal{L}^+(\mathcal{X}) \cup (\text{posi}\,\mathcal{A} + \mathcal{L}^+(\mathcal{X})). \tag{1.9}$$

The rightmost expression follows from Equation (1.3) and the fact that $\mathcal{L}^+(\mathcal{X})$ is already a convex cone.

An important result links the natural and the least committal coherent extensions:

Theorem 1.1 *The natural extension $\mathcal{E}(\mathcal{A})$ of $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ coincides with its least committal coherent extension $\bigcap \mathbb{D}_{\mathcal{A}}$ if and only if $\mathcal{A}$ avoids partial loss.*

Proof. By construction, the natural extension $\mathcal{E}(\mathcal{A})$ must be included in any coherent extension, if they exist, as they must satisfy Criteria (1.1), (1.2), and (1.4): it is therefore the least committal one if it is coherent itself. This is the case if and only if it also satisfies (1.5). From Equation (1.9) we see that $\mathcal{E}(\mathcal{A})$'s pointwise smallest gambles lie in posi $\mathcal{A}$ or $\mathcal{L}^+(\mathcal{X})$, which proves the necessary equivalence of $\mathcal{A}$ avoiding partial loss, i.e., posi $\mathcal{A} \cap \mathcal{L}^-(\mathcal{X}) = \emptyset$, and $\mathcal{E}(\mathcal{A}) \cap \mathcal{L}^-(\mathcal{X}) = \emptyset$. □

Natural extension is the prime tool for *deductive inference* in desirability: given an initial assessment, it allows us to straightforwardly deduce which gambles must also be desirable in order to satisfy coherence, but makes no further commitments.

Theory of desirable gambles: another view on natural extension

- In the literature, the natural extension within the theory of desirable gambles and its characterisation theorem are not always defined directly through the posi operator, see e.g.:

International Journal of Approximate Reasoning 53 (2012) 363–395

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Exchangeability and sets of desirable gambles

Gert de Cooman, Erik Quaeghebeur *

Ghent University, SYSTeMS Research Group, Technologiepark – Zwijnaarde 914, 9052 Zwijnaarde, Belgium

A R T I C L E I N F O

Article history:
Available online 13 December 2010

Keywords:
Sets of desirable gambles

A B S T R A C T

Sets of desirable gambles constitute a quite general type of uncertainty model with an interesting geometrical interpretation. We give a general discussion of such models and their rationality criteria. We study exchangeability assessments for them, and prove counterparts of de Finetti's Finite and Infinite Representation Theorems. We show that the finite representation in terms of count vectors has a very nice geometrical interpretation, and

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I will try to convince you that , from a logical pow, what is done e.g. here is the right approach.

their rationality criteria. We study exchangeability assessments for them, and prove counterparts of de Finetti's Finite and Infinite Representation Theorems. We show that the finite representation in terms of count vectors has a very nice geometrical interpretation, and

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Theory of desirable gambles: another view on natural extension

- First, remember that $(\mathbf{C}(\mathscr{L}) \cup \{\mathscr{L}\}, \subseteq)$ is a dually atomic complete lattice. Thus , by putting all things together, we have that:

- *Theorem*: The operator $\mathscr{E} : \wp(\mathscr{L}) \to \wp(\mathscr{L})$ defined as $\mathscr{E}(\mathscr{A}) := \bigcap \{\mathscr{K} \in \mathbf{C}(\mathscr{L}) \mid \mathscr{A} \subseteq \mathscr{K}\}$ is a closure operator such that $\mathfrak{C}_{\mathscr{E}} \backslash \{\mathscr{L}\} = \mathbf{C}(\mathscr{L})$. Moreover it holds that
$$\mathscr{E}(\mathscr{A}) = \bigcap \{\mathscr{K} \in \mathbf{M}(\mathscr{L}(\Omega)) \mid \mathscr{A} \subseteq \mathscr{K}\}.$$

Theory of desirable gambles: another view on natural extension

Theorem 1 (Natural extension). *Let $\mathscr{K}$ be a linear subspace of $\mathscr{G}(\Omega)$ and let $\mathscr{C} \subset \mathscr{K}$ be a convex cone containing the zero gamble 0. Consider an assessment $\mathscr{A} \subseteq \mathscr{K}$, and define its $(\mathscr{K},\mathscr{C})$-natural extension:*[4]

$$\mathscr{E}_{(\mathscr{K},\mathscr{C})}(\mathscr{A}) := \bigcap\{\mathscr{R} \in \mathbb{D}_{(\mathscr{K},\mathscr{C})}(\Omega) : \mathscr{A} \subseteq \mathscr{R}\}. \tag{4}$$

Then the following statements are equivalent:

(i) *$\mathscr{A}$ avoids non-positivity relative to $(\mathscr{K},\mathscr{C})$;*
(ii) *$\mathscr{A}$ is included in some set of desirable gambles that is coherent relative to $(\mathscr{K},\mathscr{C})$;*
(iii) *$\mathscr{E}_{(\mathscr{K},\mathscr{C})}(\mathscr{A}) \neq \mathscr{K}$;*
(iv) *the set of desirable gambles $\mathscr{E}_{(\mathscr{K},\mathscr{C})}(\mathscr{A})$ is coherent relative to $(\mathscr{K},\mathscr{C})$;*
(v) *$\mathscr{E}_{(\mathscr{K},\mathscr{C})}(\mathscr{A})$ is the smallest set of desirable gambles that is coherent relative to $(\mathscr{K},\mathscr{C})$ and includes $\mathscr{A}$.*

[3] We require that $\mathscr{C}$ should be *strictly included* in $\mathscr{K}$ ($\mathscr{C} \neq \mathscr{K}$) because otherwise the ordering $\succeq$ would be trivial: we would have that $f \succeq g$ for all $f,g \in \mathscr{K}$.
[4] As usual, in this expression, we let $\bigcap \emptyset = \mathscr{K}$.

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When any (and hence all) of these equivalent statements hold, then

$$\mathscr{E}_{(\mathscr{K},\mathscr{C})}(\mathscr{A}) = \text{posi}(\mathscr{K}_{\succ 0} \cup \mathscr{A}). \tag{5}$$

This shows that if we have an assessment $\mathscr{A}$ with a finite description, we can represent its natural extension on a computer by storing a finite description of its extreme rays.

Theory of desirable gambles: the case of $\mathbb{R}^n$

- *Corollary*: Let $\mathscr{A} \subseteq \mathbb{R}^n$ be a set of assessments. Then, for every $g \in \mathbb{R}^n$ the following are equivalent
    1. $g \notin \mathscr{E}(\mathscr{A})$
    2. there is $\mathscr{K} \in \mathbf{C}(\mathbb{R}^n)$ such that $\mathscr{A} \subseteq \mathscr{K}$ but $g \notin \mathscr{K}$
    3. there is $\mathscr{K} \in \mathbf{M}(\mathbb{R}^n)$ such that $\mathscr{A} \subseteq \mathscr{K}$ but $g \notin \mathscr{K}$.

The equivalence between 1 and 3 is a version of the hyperplane separation theorem but for maximal coherent sets. Recall that the latter coincide with semispaces at the origin (i.e. convex sets $\mathscr{K} \subseteq \mathbb{R}^n$ without the origin 0 and such that either $g \in \mathscr{K}$ or $-g \in \mathscr{K}$ for each $g \in \mathbb{R}^n$) that contain the positive orthant $(\mathbb{R}^n)^{>}$.

Theory of desirable gambles: natural extension

- *Definition*: Given a set of assessments $\mathscr{A} \subseteq \mathscr{L}(\Omega)$, its **natural extension** is the set

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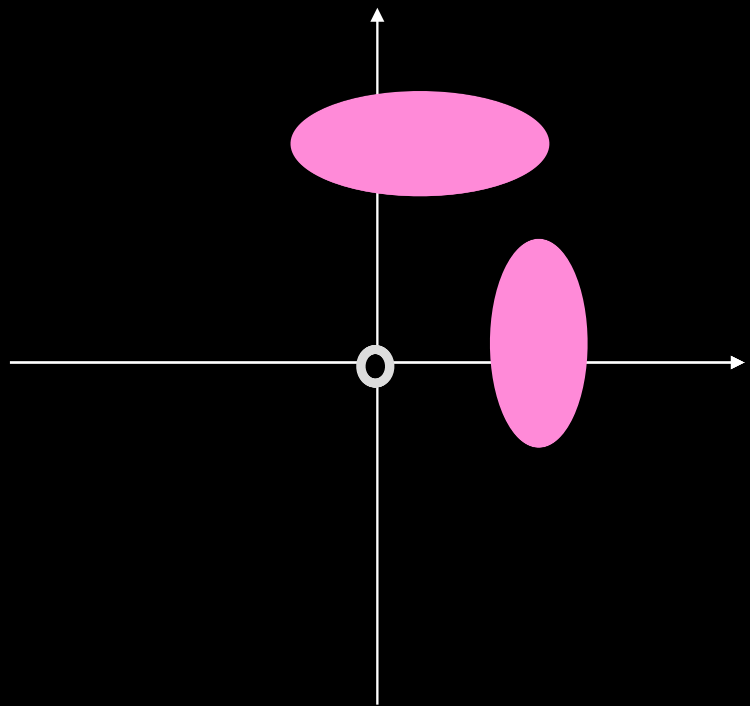

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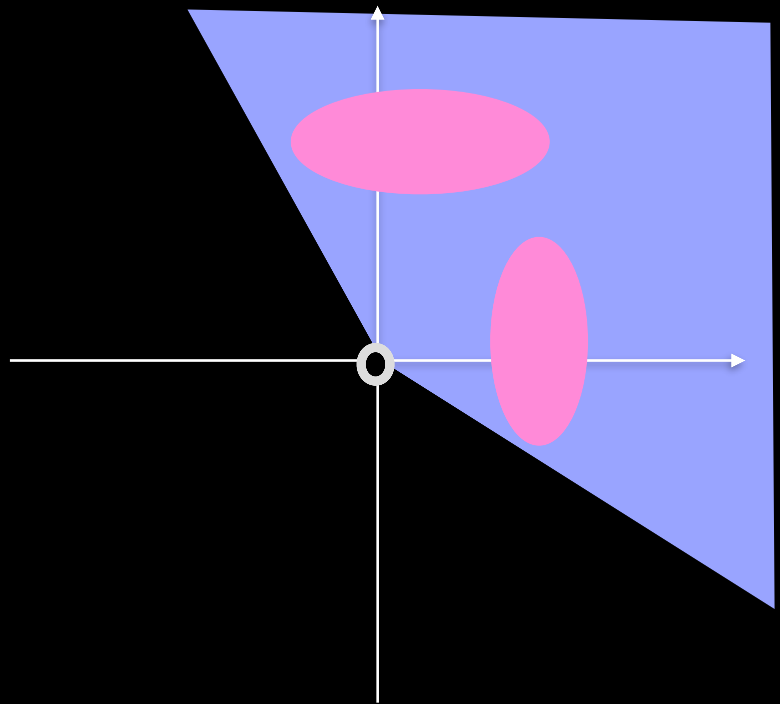

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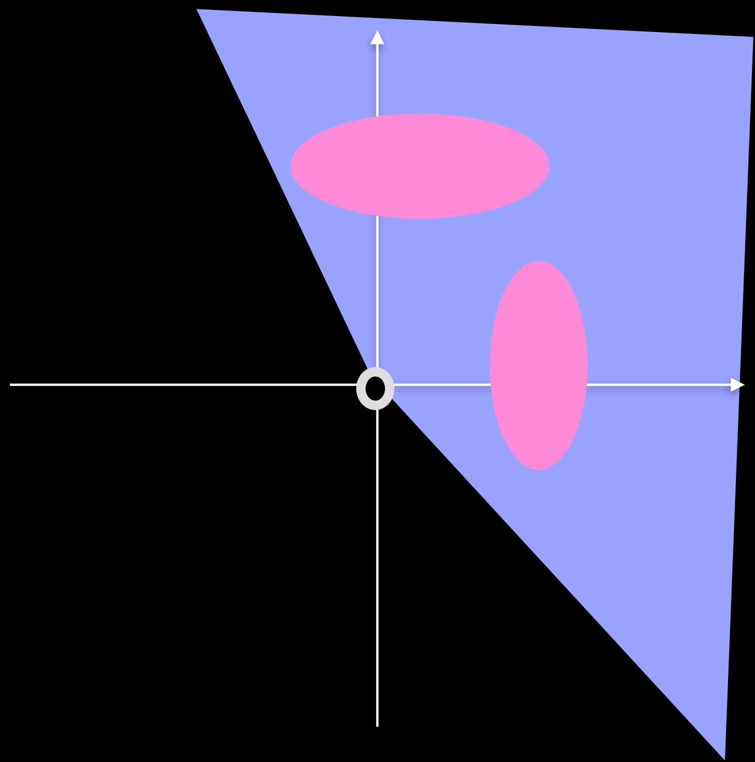

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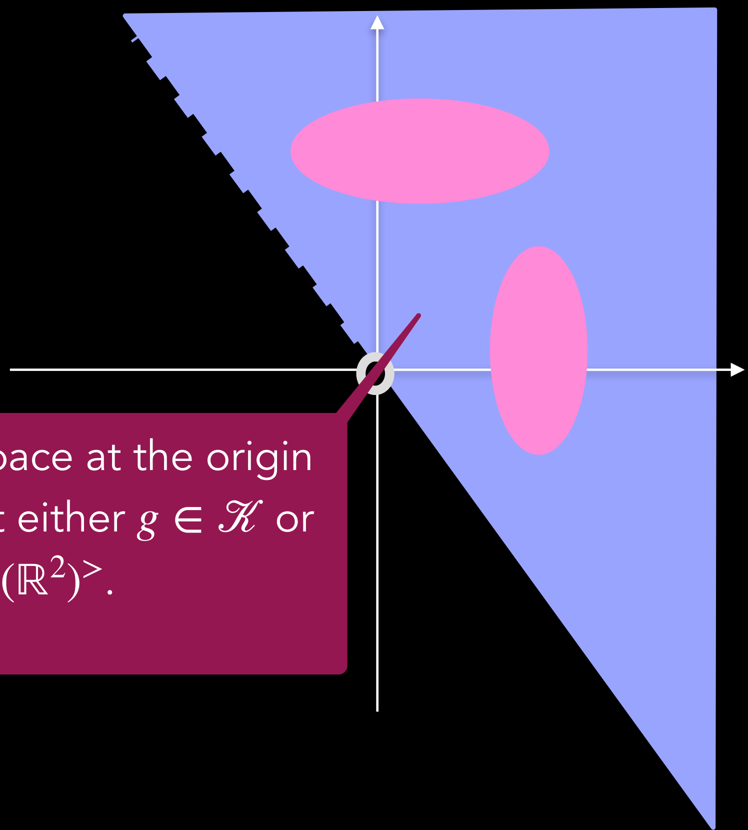

A maximal coherent set of desirable gambles, aka a semispace at the origin (i.e. convex sets $\mathscr{K} \subseteq \mathbb{R}^2$ without the origin 0 and such that either $g \in \mathscr{K}$ or $-g \in \mathscr{K}$ for each $g \in \mathbb{R}^2$) that contain the positive orthant $(\mathbb{R}^2)^>$.

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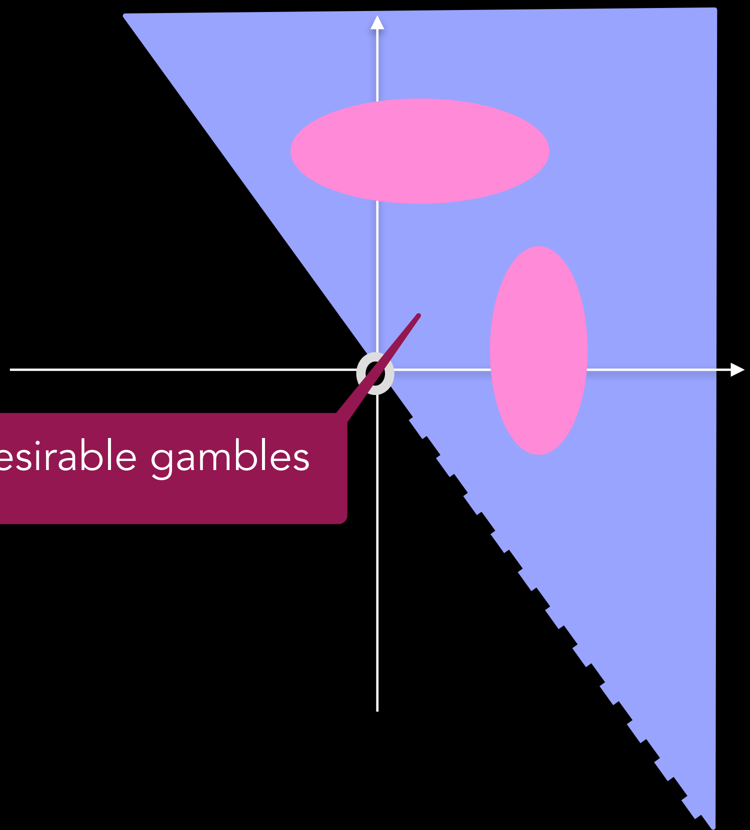

Another maximal coherent set of desirable gambles

Theory of desirable gambles: natural extension

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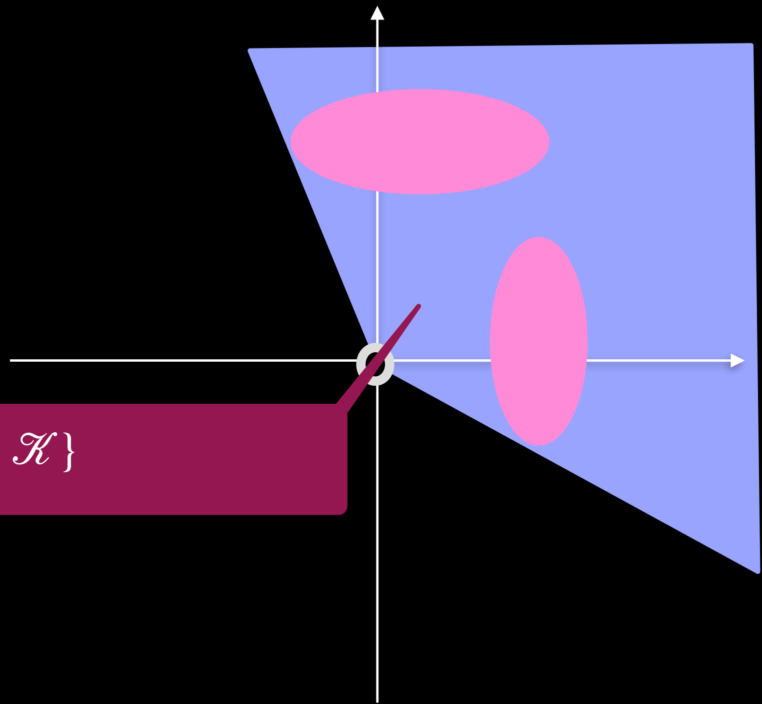


$$\mathscr{E}(\mathscr{A}) := \bigcap \{\mathscr{K} \in \mathbf{C}(\mathscr{L}(\Omega)) \mid \mathscr{A} \subseteq \mathscr{K}\}$$

Theory of **almost** desirable gambles

- In the context of the theory of almost desirable gambles the two views actually coincide.
- Definition: A set $\mathscr{K} \subseteq \mathscr{L}(\Omega)$ is coherent if it satisfies

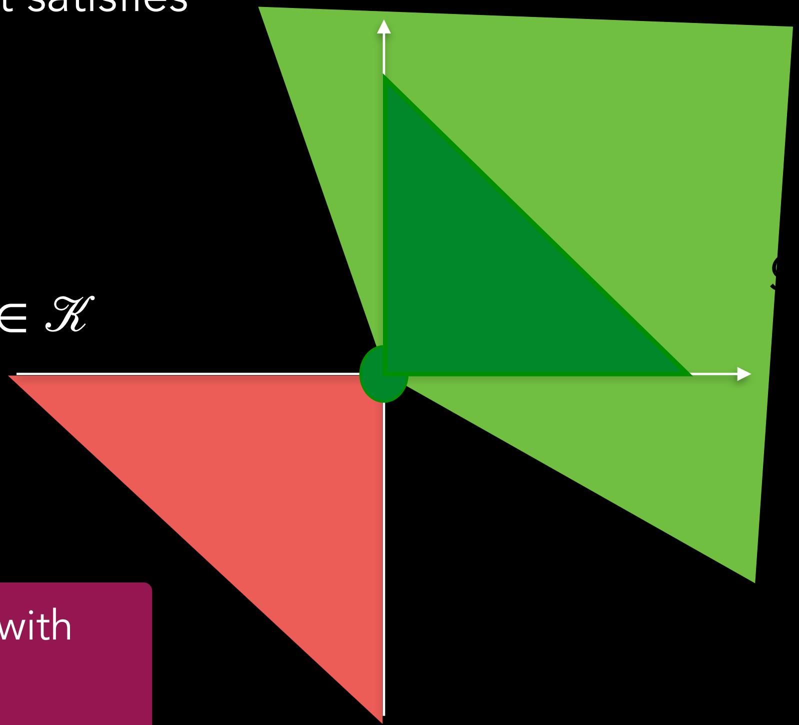
  (APG) $\mathscr{L}^{>} \subseteq \mathscr{K}$

  (PS) $\lambda\mathscr{K} \subseteq \mathscr{K}$, for $\lambda > 0$

  (ADD) $\mathscr{K} + \mathscr{K} \subseteq \mathscr{K}$

  (CL) $f \in \mathscr{K}$, if $\exists \epsilon \in (0,1) \forall k > 0 : f + \epsilon^k \in \mathscr{K}$

  (ASL) $\mathscr{L}^{-} \cap \mathscr{K} = \emptyset$



Given the preceding conditions, (ASL) is equivalent with (ANU) $-1 \notin \mathscr{K}$

Theory of **almost** desirable gambles

Similarly to what done before, by

- $\mathbf{C}_a(\mathscr{L}(\Omega))$ we denote the collection of all coherent sets of almost desiderable gambles,
- $\mathbf{M}_a(\mathscr{L}(\Omega))$ we denote the collection of all maximal coherent sets of almost desiderable gambles.

Notice that maximal coherent sets are simply the closed halfspaces containing the positive orthant and the origin in the boundary.

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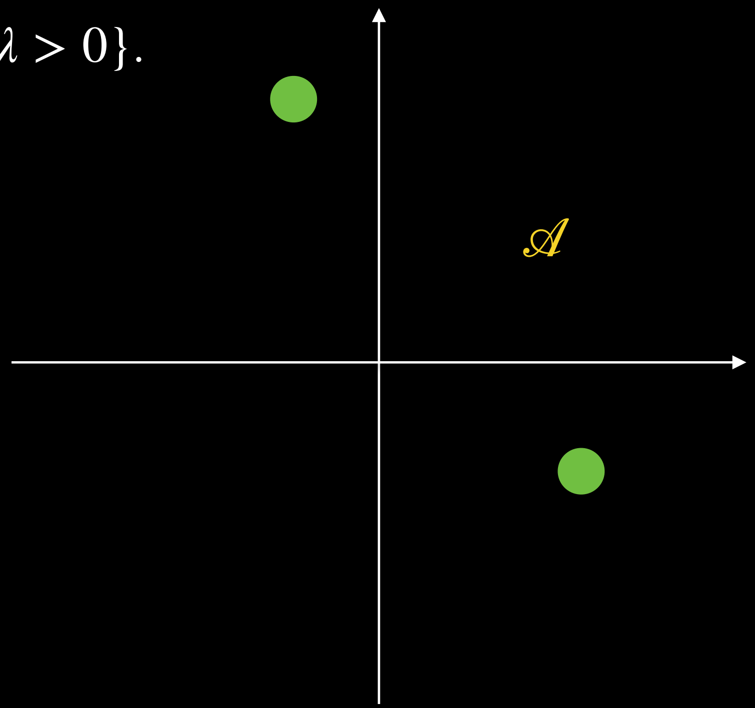
Theory of **almost** desirable gambles

- *Definition*: Given a set of assessments $\mathscr{A} \subseteq \mathscr{L}(\Omega)$, its **natural extension** is the set

$$\mathscr{E}_a(\mathscr{A}) := \mathrm{cl}(\mathrm{posi}(\mathscr{A} \cup \mathscr{L}^{>})),$$

where $\mathrm{posi}(X) := \{\mu f + \lambda g \mid f, g \in X, \text{ and } \mu, \lambda > 0\}$.



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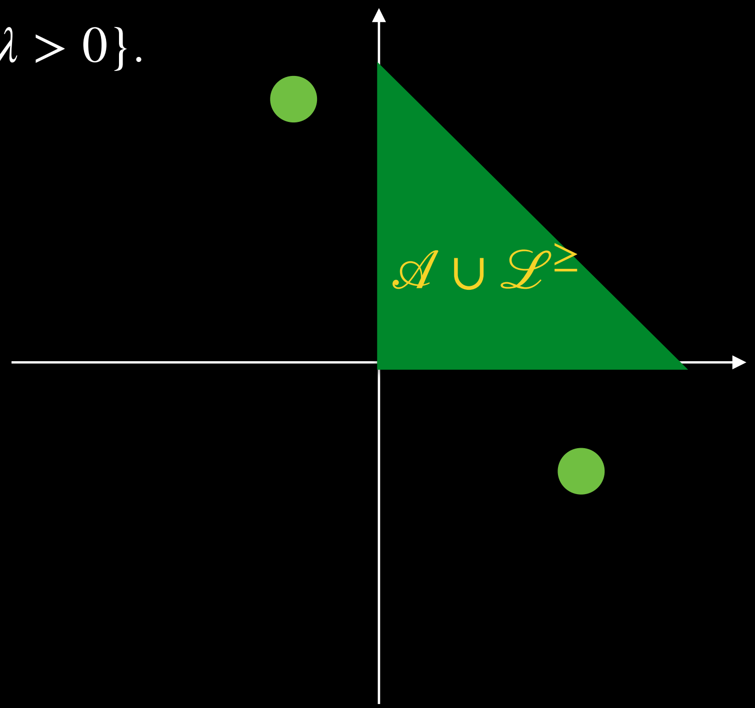

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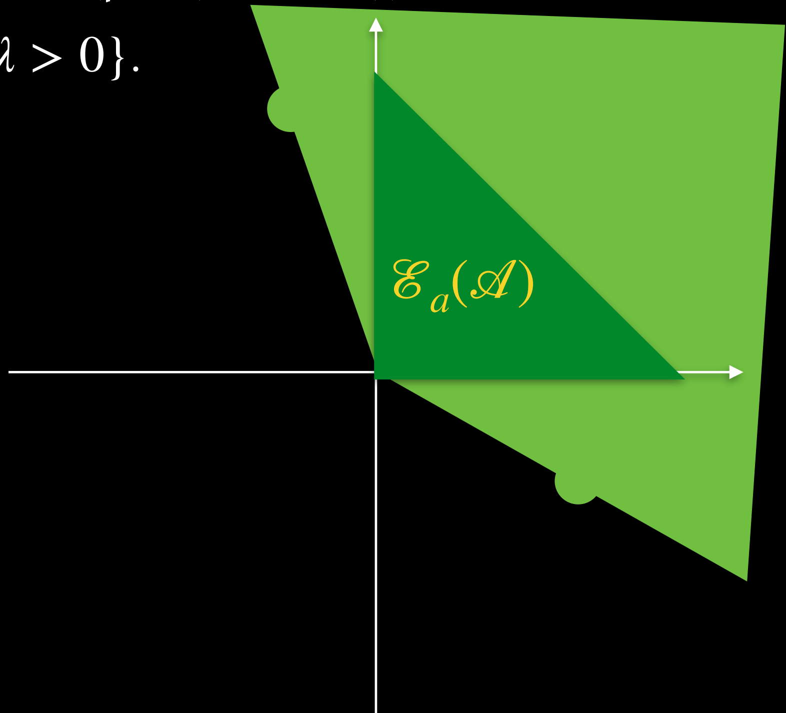

Theory of **almost** desirable gambles: characterisation

- *Facts*: The following properties hold
  1. The lattice $(\mathbf{C}_a(\mathscr{L}) \cup \{\mathscr{L}\}, \subseteq)$ is complete and dually atomic, and the collection of its dual atoms coincides with $\mathbf{M}_a(\mathscr{L})$, i.e. the collection of all closed halfspaces containing the positive orthant and the origin in the boundary.
  2. The natural extensions operator $\mathscr{E}_a(\cdot) := \mathrm{cl}(\mathrm{posi}(\cdot \cup \mathscr{L}^{\geq})) : \wp(\mathscr{L}) \to \wp(\mathscr{L})$ is a closure operator, and the collection of its closed sets coincides with $\mathbf{C}_a(\mathscr{L}) \cup \{\mathscr{L}\}$.

Theory of **almost** desirable gambles: characterisation

Hence, from the previous facts, the definition of coherence in the theory of almost desirable gambles and the properties of posets, closure operators and dually atomic complete lattices, we obtain that:

- *Corollary*: Given a set of assessments $\mathscr{A} \subseteq \mathscr{L}(\Omega)$, the following are equivalent
  1. $\mathscr{E}_a(\mathscr{A}) \in \mathbf{C}_a(\mathscr{L})$,
  2. $\mathscr{E}_a(\mathscr{A}) = \bigcap \{\mathscr{K} \in \mathbf{C}_a(\mathscr{L}) \mid \mathscr{A} \subseteq \mathscr{K}\}$,
  3. $\mathscr{E}_a(\mathscr{A}) = \bigcap \{\mathscr{K} \in \mathbf{M}_a(\mathscr{L}) \mid \mathscr{A} \subseteq \mathscr{K}\}$,
  4. $-1 \notin \mathscr{E}_a(\mathscr{A})$
  5. $\mathscr{E}_a(\mathscr{A}) \neq \mathscr{L}$

This is essentially provides a variant of the classical separation theorem for closed convex sets.

To sum up

	TDG				TADG
defining coherence condition	APL		ASQ		ASL
consequence (closure) operator	posi	Cl on coherence	posi	Cl on coherence	cl-posi
corresponding theories (closed sets) are all coherent	no	yes	no	yes	yes
complete lattice C + L dually atomic	no	no	no	yes	yes

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To sum up

If you add 0 to a semispace at the origin K, you get a coherent dual atom, and K is not the intersection of any coherent dual atom extending it

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To sum up

The difference between deductive closure and coherence, from a logical perspective, is related to the phenomenon of *paraconsistency*

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Trying (again) to make a bit of order

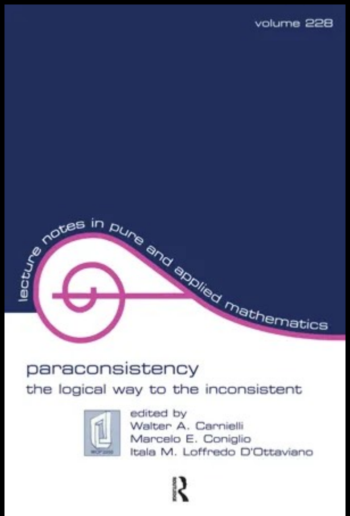
Belief models: An order-theoretic investigation

Gert de Cooman (`gert.decooman@ugent.be`)
Universiteit Gent, Onderzoeksgroep SYSTeMS, Technologiepark – Zwijnaarde 914, 9052 Zwijnaarde, Belgium

September 7, 2010

Abstract. I show that there is a common order-theoretic structure underlying many of the models for representing beliefs in the literature. After identifying this structure, and studying it in some detail, I argue that it is useful. On the one hand, it can be used to study the relationships between several models for representing beliefs, and I show in particular that the model based on classical propositional logic can be embedded in that based on the theory of coherent lower previsions. On the other hand, it can be used to generalise the coherentist study of belief dynamics (belief expansion and revision) by using an abstract order-theoretic definition of the belief spaces where the dynamics of expansion and revision take place. Interestingly, many of the existing results for expansion and revision in the context of classical propositional logic can still be proven in this much more abstract setting, and therefore remain valid for many other belief models, such as those based on imprecise probabilities.

Keywords: Belief model, belief revision, classical propositional logic, imprecise probability, order theory, possibility measure, system of spheres



Belief models

- *Definition*: A **belief structure** is a quadruple $(A, \leq, \mathsf{cl}, C)$ where
  - $(A, \leq)$ is a complete lattice, and $a \in A$ is called a belief model,
  - $\mathsf{cl} : A \to A$ a closure operator over $(A, \leq)$, and a non-empty $C \subseteq \mathfrak{C}_{\mathsf{cl}}$, called a coherence predicate, such that
    1. it is reverse compatible with cl, that is: if $a \leq b$ and $\mathsf{cl}(b) \in C$, then $\mathsf{cl}(a) \in C$, meaning in particular that $\mathsf{cl}(0) \in C$,
    2. $(C \cup \{1\}, \leq)$ is a closure system, meaning in particular that C is closed under non-empty infima (i.e. $\bigwedge B \in C$ for every $\emptyset \neq B \subseteq C$)

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  If $C \cup \{1\} = \mathfrak{C}_{\mathsf{cl}}$, the belief structure is called **classical**, and **paraconsistent** otherwise.

The idea is that a belief structure is classical whenever the deductive closure of every inconsistent belief model is trivial. Paraconsistent belief structures are structures in which, for closed models, being inconsistent is not tantamount to being trivial. Stated otherwise, inconsistency is not 'explosive'.

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  A classical belief structure for which $(C \cup \{1\}, \leq)$ is dually atomic is said to be **strong**.

Because of this, we will actually be able to provide prove a strong, "natural" completeness theorem in term of probabilistic semantics

To sum up again

	TDG				TADG
defining coherence condition	APL		ASQ		ASL
closure operator	posi	Cl on coherence	posi	Cl on coherence	cl-posi
belief structure	yes	yes	yes	yes	yes
classical	no	yes	no	yes	yes (and thus cl-posi is Cl on coherence)
strong	no	no	no	yes	yes

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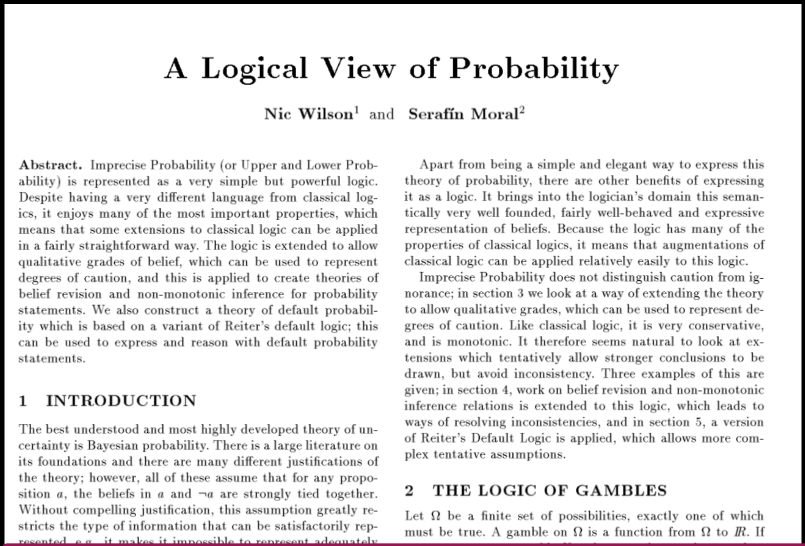
Desirability as a (abstract) logic

Complete probabilistic semantics for desirability as logic: first attempts



A Logical View of Probability

Nic Wilson[1] and Serafín Moral[2]

Abstract. Imprecise Probability (or Upper and Lower Probability) is represented as a very simple but powerful logic. Despite having a very different language from classical logics, it enjoys many of the most important properties, which means that some extensions to classical logic can be applied in a fairly straightforward way. The logic is extended to allow qualitative grades of belief, which can be used to represent degrees of caution, and this is applied to create theories of belief revision and non-monotonic inference for probability statements. We also construct a theory of default probability which is based on a variant of Reiter's default logic; this can be used to express and reason with default probability statements.

1 INTRODUCTION

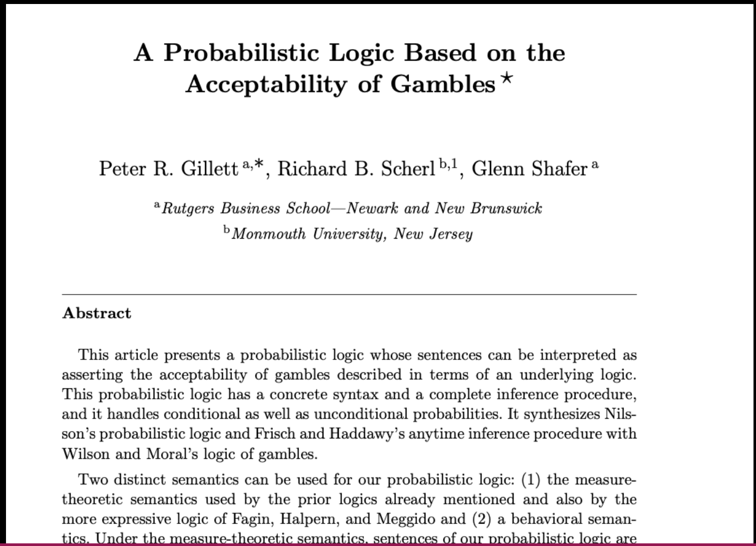
The best understood and most highly developed theory of uncertainty is Bayesian probability. There is a large literature on its foundations and there are many different justifications of the theory; however, all of these assume that for any proposition a, the beliefs in a and $\neg a$ are strongly tied together. Without compelling justification, this assumption greatly restricts the type of information that can be satisfactorily represented, e.g., it makes it impossible to represent adequately

Apart from being a simple and elegant way to express this theory of probability, there are other benefits of expressing it as a logic. It brings into the logician's domain this semantically very well founded, fairly well-behaved and expressive representation of beliefs. Because the logic has many of the properties of classical logics, it means that augmentations of classical logic can be applied relatively easily to this logic.

Imprecise Probability does not distinguish caution from ignorance; in section 3 we look at a way of extending the theory to allow qualitative grades, which can be used to represent degrees of caution. Like classical logic, it is very conservative, and is monotonic. It therefore seems natural to look at extensions which tentatively allow stronger conclusions to be drawn, but avoid inconsistency. Three examples of this are given; in section 4, work on belief revision and non-monotonic inference relations is extended to this logic, which leads to ways of resolving inconsistencies, and in section 5, a version of Reiter's Default Logic is applied, which allows more complex tentative assumptions.

2 THE LOGIC OF GAMBLES

Let Ω be a finite set of possibilities, exactly one of which must be true. A gamble on Ω is a function from Ω to $I\!R$. If



A Probabilistic Logic Based on the Acceptability of Gambles ★

Peter R. Gillett [a,*], Richard B. Scherl [b,1], Glenn Shafer [a]

[a] *Rutgers Business School—Newark and New Brunswick*
[b] *Monmouth University, New Jersey*

Abstract

This article presents a probabilistic logic whose sentences can be interpreted as asserting the acceptability of gambles described in terms of an underlying logic. This probabilistic logic has a concrete syntax and a complete inference procedure, and it handles conditional as well as unconditional probabilities. It synthesizes Nilsson's probabilistic logic and Frisch and Haddawy's anytime inference procedure with Wilson and Moral's logic of gambles.

Two distinct semantics can be used for our probabilistic logic: (1) the measure-theoretic semantics used by the prior logics already mentioned and also by the more expressive logic of Fagin, Halpern, and Meggido and (2) a behavioral semantics. Under the measure-theoretic semantics, sentences of our probabilistic logic are

Wilson & Moral introduced a (semi-formal) logical calculus for *almost* desirability and show finite completeness with respect to probabilistic semantics.

Later Gillet, Scherl & Shafer adapted the calculus to desirability, incorporated the conditioning operation and proved finite completeness.

In this talk we are only interested, if not explicitly stated, to sentential / propositional logic systems.

To get this just fix a set of rules schemas

A bit more structure, please

- We are usually interested in the "meaning" of connectives, and in characterising the associated formal deductive inferences (reasoning).
- So you usually need first to fix a set of connectives L and a language (an algebra) over (of signature) L….
- … and then define a consequence relation over the set of formulas.

In the abstract algebraic perspective, a **sentential logic** is a consequence system (Fm, $\vdash$) given by the absolutely free L-algebra Fm generated by a set of propositional variables V (i.e the smallest L-algebra containing V and such that for every other L -algebra A, a map h : V $\rightarrow A$ can uniquely be extended to a homomorphism h' : Fm $\rightarrow$ A) and where $\vdash$ satisfies *structurality*: if $\Gamma \vdash \varphi$ then $h(\Gamma) \vdash h(\varphi)$, for every substitution (endomorphism) h : Fm $\rightarrow$ Fm.

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- … and then define a consequence relation over the set of formulas.
    - *Syntactic* characterisation: provide a list of rules $\mathfrak{R}$, e.g. Hilbert sytle or Gentzen style, via structural ones, corresponding to the abstract properties of a consequence relation, and specific rules for connectives
    - *Semantic* characterisation: make reference to "something else", external.

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Typically one is interested in having a relation semantics $\mathfrak{S}$ (e.g. Kripke/possible world semantics), as it provides an intuitive interpretation of the logic and a means to obtain information about it, and particularly obtaining a **completeness results stating that** $\vdash_{\mathfrak{R}} = \vdash_{\mathfrak{S}}$.

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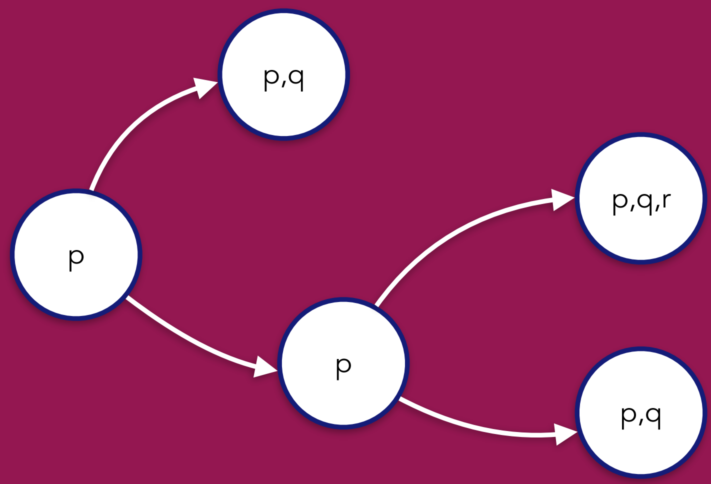
A bit more structure, please

A poset $(W, \leq)$ equipped with a valuation function $\mathfrak{s}_w$ for each $w \in W$ such that if $w \leq w'$ then $\mathfrak{s}_w \subseteq \mathfrak{s}_{w'}$.



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Many logics are closely related to a classes of algebraic structures, e.g. Boolean algebras (with operators), or Heyting algebras. Now associating algebraic models to propositional logics is often achieved by an easy transcription of the syntactic specifications of such logics, e.g. through the associated Lindenbaum–Tarski algebras or through a transcription of a Gentzen-style calculus. As a consequence, semantic modelling by such algebras is often not far removed from the syntactic treatment of the logics.

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  - *Semantic* characteris

Consider classical logic $(\mathscr{L}, \vdash_{CL})$.

Fix any theory (closed set of formulas) Γ, and define the congruence θ_Γ such thar $\varphi\theta_\Gamma\psi$ iff $(\Gamma, \varphi \vdash_{CL} \psi$ and $\Gamma, \psi \vdash_{CL} \varphi)$.

Then the quotient algebra $\mathbf{Fm}/\theta_\Gamma$ is a boolean algebra (check), and given the valuation

$$\mathfrak{s}_\Gamma : \varphi \mapsto \begin{cases} a & \text{if } \varphi/\theta_\Gamma \in \Gamma/\theta_\Gamma \\ r & \text{else} \end{cases}$$

This is a (canonical) interpretation of $(\mathscr{L}, \vdash_{CL})$ such that if $\Gamma \nvdash_{CL} \varphi$ then $\Gamma \nvdash_{\mathfrak{s}_\Gamma} \varphi$.

It is then immediate to check that $\vdash_{CL} = \vdash_{\mathfrak{S}} := \bigcap_{\Gamma \subseteq Fm} \vdash_{\mathfrak{s}_\Gamma}$

Many logics are closely related to a classes of algebraic structures, e.g. Boolean algebras (with operators), or Heyting algebras. Now associating algebraic models to propositional logics is often achieved by an easy transcription of the syntactic specifications of such logics, e.g. through the associated Lindenbaum–Tarski algebras or through a transcription of a Gentzen-style calculus. As a consequence, semantic modelling by such algebras is often not far removed from the syntactic treatment of the logics.

What is the relation between this two views?

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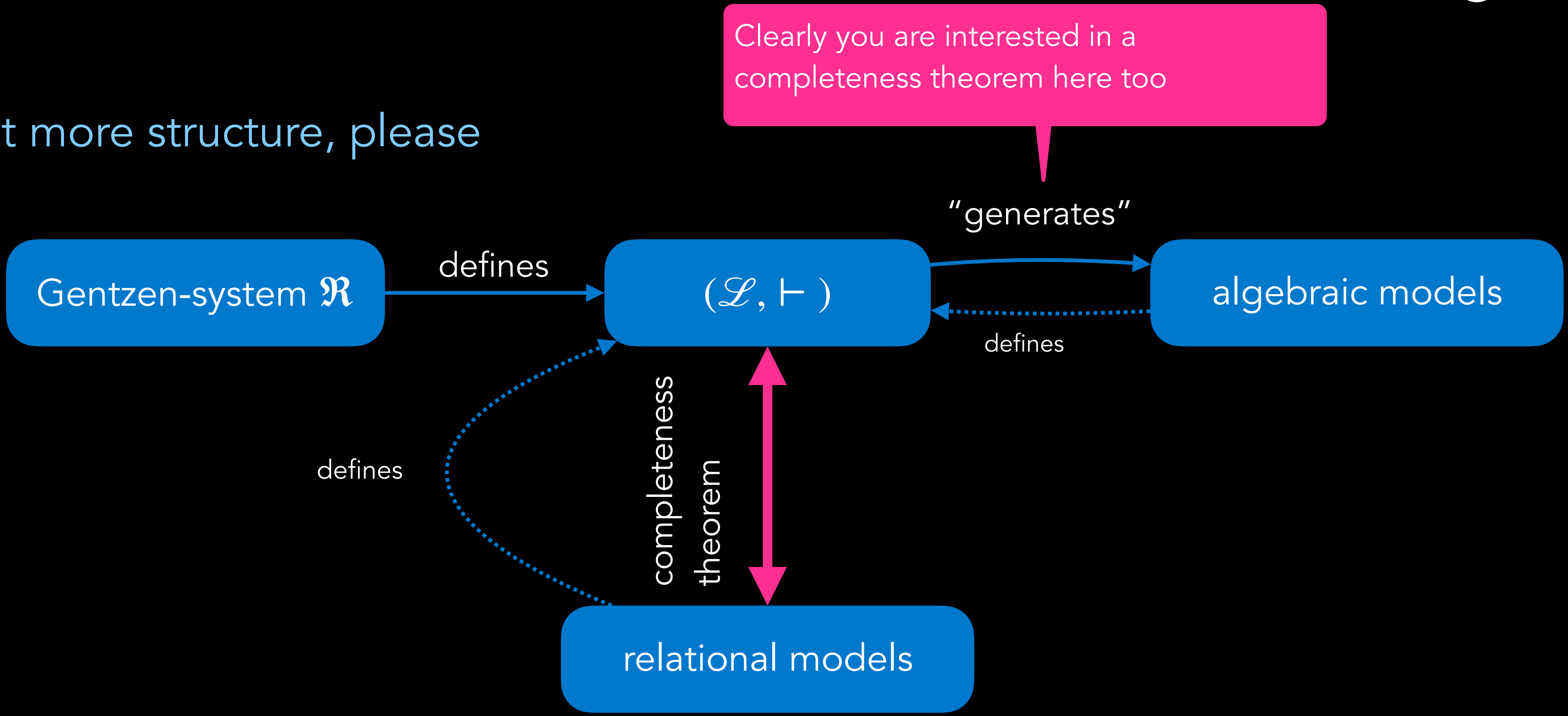
- … and then define a consequence relation over the set of formulas.
  - *Syntactic* characterisation: provide a list of rules $\mathfrak{R}$, e.g. Hilbert sytle or Gentzen style, via structural ones, corresponding to the abstract properties of a consequence relation, and specific rules for connectives
  - *Semantic* characterisation: make reference to "something else", external.

Typically one is interested in having a relation semantics $\mathfrak{S}$ (e.g. Kripke/possible world semantics), as it provides an intuitive interpretation of the logic and a means to obtain information about it, and particularly obtaining a **completeness results stating that** $\vdash_{\mathfrak{R}} = \vdash_{\mathfrak{S}}$.

A bit more structure, please

A bit more structure, please



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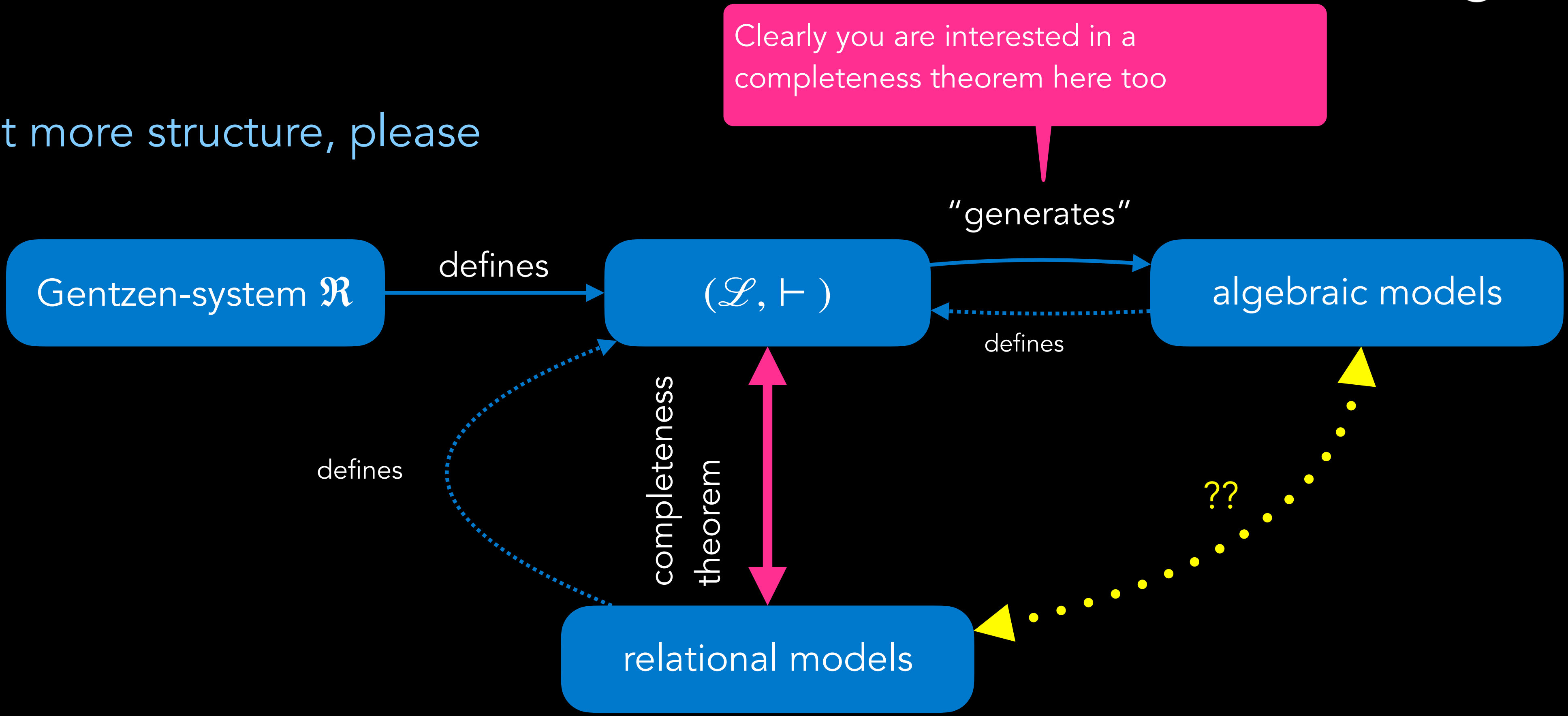
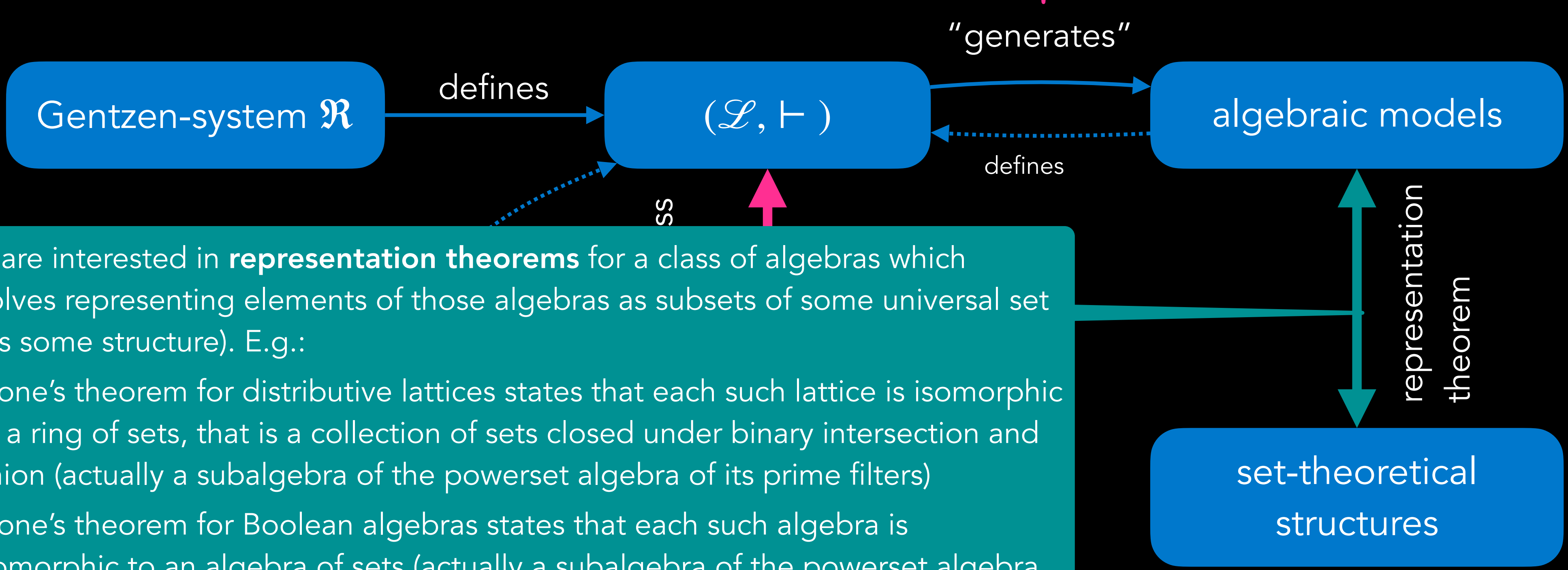


A bit more structure, please

Clearly you are interested in a completeness theorem here too



We are interested in **representation theorems** for a class of algebras which involves representing elements of those algebras as subsets of some universal set (plus some structure). E.g.:

- Stone's theorem for distributive lattices states that each such lattice is isomorphic to a ring of sets, that is a collection of sets closed under binary intersection and union (actually a subalgebra of the powerset algebra of its prime filters)
- Stone's theorem for Boolean algebras states that each such algebra is isomorphic to an algebra of sets (actually a subalgebra of the powerset algebra of its ultrafilters).

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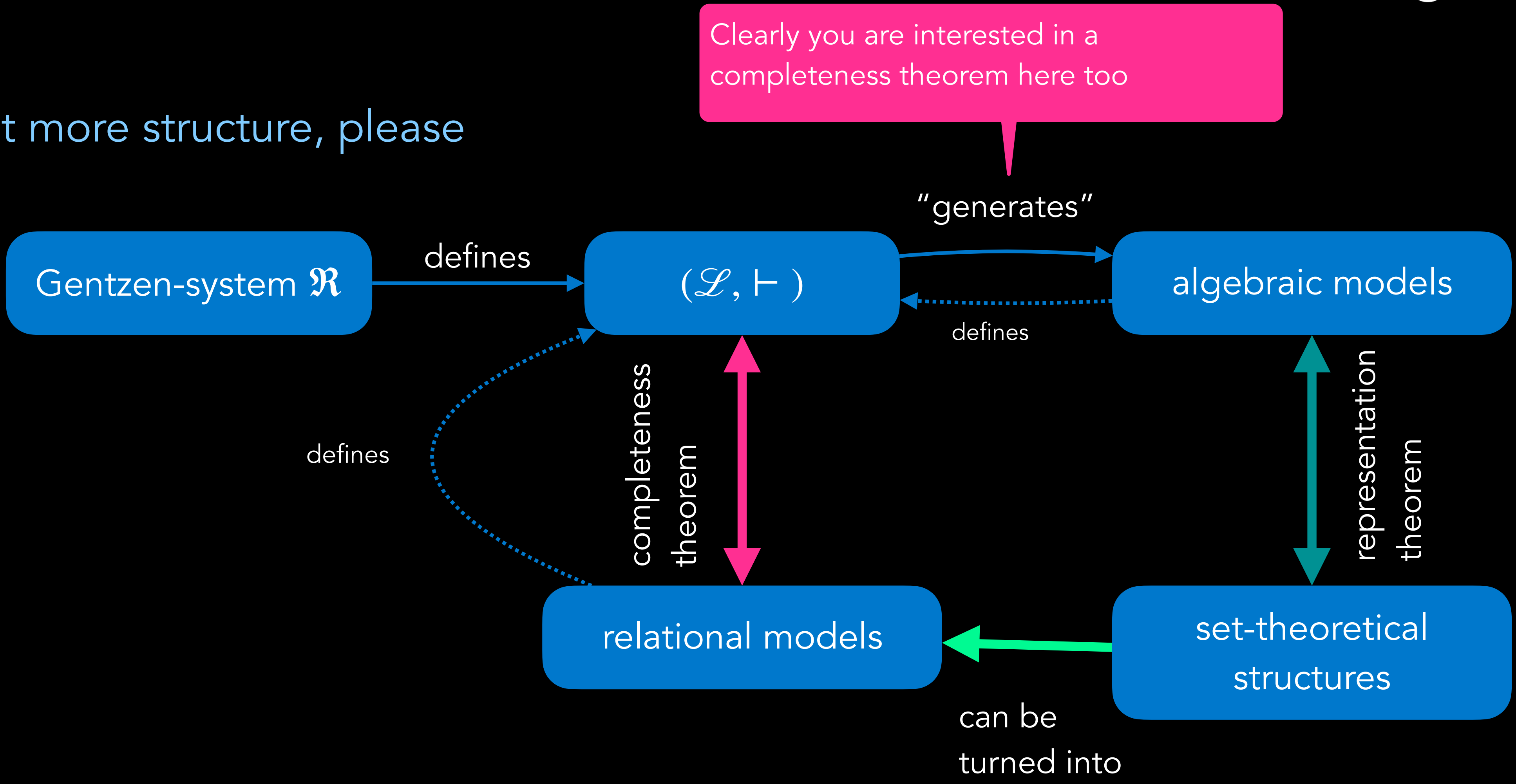
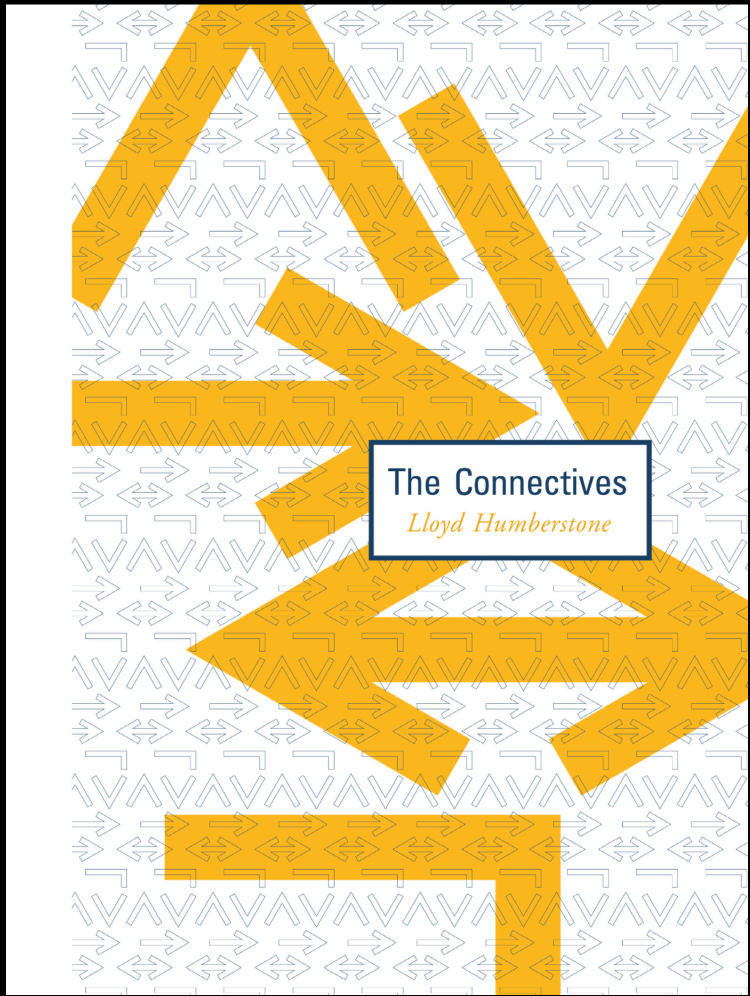
A bit more structure, please



A bit more structure, please

- So, for simplicity, from now on a (abstract) logic is some consequence system $(A, \vdash)$ where A is an algebra.
- How to assess if $(A, \vdash)$ has, say, a conjunction? Do I need to have a binary operator? And what about constant (or 0-ary operators) such as the falsum? How it relates with coherence (consistency)? And negation?
- More abstract perspective, we will ask for some structural conditions in order to state that some form over A acts as/represents a certain (well known) connective



The falsum

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The falsum

- We can imagine to dispose in our language of some specific “thing” (constant, or 0-ary connective in the underlying algebra), let us denote it by $\bot$, that enables us to speak about coherence (or consistentcy): the falsum.
- Assume we pick $\bot$ in the underlying language of a consequence system $(A, \vdash)$, we can thus readily characterise a consistency predicate $\mathsf{C}_\bot$ as follows:
  - $\mathsf{Cn}_\vdash(B) \in \mathsf{C}_\bot$ iff $B \nvdash \bot$, with $B \subseteq A$,

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Clearly $\mathsf{C}_\bot$ is closed under arbitrary non-empty intersections and, by exploiting monotonicity of $\mathsf{Cn}_\vdash$, it is reverse compatible with $\mathsf{Cn}_\vdash$. Hence

- *Fact*: Given a consequence system $(A, \vdash)$ and a designated element $\bot \in A$, the 4-uple $(\wp(A), \subseteq, \mathsf{Cn}_\vdash, \mathsf{C}_\bot)$ is a belief structure.

The falsum

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  - $Cn_\vdash(B) \in C_\bot$ iff $B \nvdash \bot$, with $B \subseteq A$,

Thus, we say that, given a consequence system $(A, \vdash)$, a belief structure $(\wp(A), \subseteq, Cn_\vdash, C)$ is **definable** if there is a designated element $\bot \in A$ such that $C = C_\bot$.

The falsum

- *Definition*: Let $(A, \vdash)$ be a consequence system. We say that $\bot \in A$ is a **falsum** for $(A, \vdash)$ if the following principle holds, for every $b \in A$, and every $\Gamma \subseteq A$
  - If $\Gamma \vdash \bot$, then $\Gamma \vdash b$ (ex-falso sequitur quod libet / ⊥-elimination)

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From the definitions, we get that:

- *Fact*: Given a consequence system $(A, \vdash)$, a definable belief structure $(\wp(A), \subseteq, \mathrm{Cn}_{\vdash}, \mathrm{C}_{\bot})$ is classical if and only if $\bot \in A$ is a falsum.

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Calculi for desirabilit(ies)

A logical calculus for desirability over $\mathbb{R}^n$: sequents

- A *sequent* is a pair (Γ, g), also written $\Gamma \rhd g$, where Γ is a set of gambles over $\mathbb{R}^n$, and g is a gamble in $\mathbb{R}^n$.
- We read a sequent $\Gamma \rhd g$ as "***whenever Alice accepts Γ, she also accepts g***"

A logical calculus for desirability over $\mathbb{R}^n$: structural rules

- We set a family of rules to be sure that the induced relation is a consequence relation

$$\frac{}{\Gamma \rhd g}\,R \quad \text{for } g \in \Gamma$$

$$\frac{\Phi \rhd g}{\Gamma \rhd g}\,D \quad \text{for } \Phi \subseteq \Gamma$$

$$\frac{\Gamma \rhd g \qquad \Gamma, g \rhd f}{\Gamma \rhd f}\,\text{cut}$$

A logical calculus for desirability over $\mathbb{R}^n$: structural rules

- We then set a family of rules corresponding to the positive hull ones

$$\frac{}{\Gamma \triangleright g}\ \text{APG} \quad \text{for } g > 0$$

$$\frac{\Gamma \triangleright g}{\Gamma \triangleright \lambda g}\ \text{PS} \quad \text{for } \lambda > 0$$

$$\frac{\Gamma \triangleright g \qquad \Gamma \triangleright f}{\Gamma \triangleright g + f}\ \text{ADD}$$

A logical calculus for desirability over $\mathbb{R}^n$: structural rules

- And finally, we (may) add a rule stating that $\perp := 0$ is a falsum

$$\frac{\Gamma \triangleright \perp}{\Gamma \triangleright g} \; \perp\text{-el.}$$

A logical calculus for desirability over $\mathbb{R}^n$: structural rules

- All previous Gentzen-sytle rules constitute the calculus $\mathfrak{D}$ for TDG; and denote by $\mathfrak{D}^-$ the system $\mathfrak{D}$ without the rule for the elimination of the falsum.
- A sequent $\Gamma \rhd g$ is **provable in a calculus $\mathfrak{X}$**, and write $\Gamma \vdash_{\mathfrak{X}} g$, if there is a tree of finite depth such that:
  - its root is labelled by $\Gamma \rhd g$,
  - its leaves are labelled with axioms of $\mathfrak{X}$ (a rule without premisses, e.g. APG), and
  - each intermediate nodes is labelled according to the rules of $\mathfrak{X}$

Example of deduction in $\mathfrak{D}$

$$\frac{}{f_1, f_2 \triangleright f_1}\ \text{R} \qquad \frac{}{f_1, f_2 \triangleright f_2}\ \text{R}$$

$$\frac{f_1, f_2 \triangleright f_1 \qquad f_1, f_2 \triangleright f_2}{f_1, f_2 \triangleright f_1 + f_2}\ \text{ADD}$$

$$\frac{f_1, f_2 \triangleright f_1 + f_2}{f_1, f_2 \triangleright 0.7(f_1 + f_2)}\ \text{PS}$$

$$\frac{}{\triangleright 1}\ \text{APG}$$

$$\frac{f_1, f_2 \triangleright 0.7(f_1 + f_2) \qquad \triangleright 1}{f_1, f_2 \triangleright 0.7(f_1 + f_2) + 1}\ \text{ADD}$$

On simple completeness results and belief structures for TDG

- It is obvious that the following hold

> For every $\Gamma \cup \{\varphi\} \subseteq \mathscr{L}$, $\Gamma \vdash_{\mathfrak{D}} \varphi$ iff for every $\Delta \in \mathbf{M}(\mathscr{L})$, $\Gamma \subseteq \Delta$ implies $\varphi \in \Delta$

- As we know, this is not true for $\mathfrak{D}^-$
- When considering $\mathfrak{D}^-$ and the corresponding TDG, the underlying belief structure is definable and paraconsistent, whereas for $\mathfrak{D}$ it is also definable but classical.

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What about a completeness theorem but wrt a probabilistic semantics?

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What about **other forms** of desirability?

What about **almost** desirability?

- To get a calculus $\mathfrak{A}$ for the theory of almost desirability such that

For every $\Gamma \cup \{\varphi\} \subseteq \mathscr{L}$, $\Gamma \vdash_{\mathfrak{A}} \varphi$ iff for every $\Delta \in \mathbf{M}_a(\mathscr{L})$, $\Gamma \subseteq \Delta$ implies $\varphi \in \Delta$

- It is enough to add the following infinitary rule for closure and re-formulate the falsum for -1

$$\frac{\{\, \Gamma \triangleright (g + \delta^n) : n > 0 \,\}}{\Gamma \triangleright g} \quad (\delta > 0)$$

A generalised theory of desirability

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Main source……………………………………………and another approach

Proceedings of Machine Learning Research 215:25–36, 2023 ISIPTA 2023

Closure Operators, Classifiers and Desirability

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Abstract

At the core of Bayesian probability theory, or dually desirability theory, lies an assumption of linearity of the scale in which rewards are measured. We revisit two recent papers that extend desirability theory to the nonlinear case by letting the utility scale be represented either by a general closure operator or by a binary general (nonlinear) classifier. By using standard results in logic, we highlight the connection between these two approaches and show that this connection allows us to extend the separating hyper plane theorem (which is at the core of the duality between Bayesian decision theory and desirability theory) to the nonlinear case.

Keywords: closure operators, classifiers, desirability, belief structure

starts from the observation that the logical consistency of a set of linearly-desirable gambles can be checked by solving a binary linear classification problem. Then the authors extend desirability to the nonlinear case by instead considering a binary nonlinear classification problem. This framework imposes the logical constraints of desirability theory by forcing the classifier to separate the non-negative gambles (gaining money is desirable) from the negative ones (losing money is undesirable). Moreover, theoretical results and numerical algorithms are provided to learn classifiers from a dataset made of accepted and rejected gambles for three closure operators: conic hull, convex hull and the so-called orthant-hull (or monotonic-hull).

The works [15, 6] show that the previous approaches to nonlinear-desirability [16, 18] can be seen as particular cases of these formulations.

A theory of desirable things

Jasper De Bock

Abstract

Inspired by the theory of desirable gambles that is used to model uncertainty in the field of imprecise probabilities, I present a theory of desirable things. Its aim is to model a subject's beliefs about which things are desirable. What the things are is not important, nor is what it means for them to be desirable. It can be applied to gambles, calling them desirable if a subject accepts them, but it can just as well be applied to pizzas, calling them desirable if my friend Arthur likes to eat them. Other useful examples of things one might apply this theory to are propositions, horse lotteries, or preferences between any of the above. Regardless of the particular things that are considered, inference rules are imposed by means of an abstract closure operator, and models that adhere to these rules are called coherent. I consider two types of models, each of which can capture a subject's beliefs about which things are desirable: sets of desirable things and sets of desirable sets of things. A crucial result is that the latter type can be represented by a set of the former.

A generalised theory of desirability

- Consider the underlying language $\mathscr{L}$ is given by a ordered vector space, with null element 0 and (order) unit 1
- The space of gambles that should be “objectively” accepted is thus $\mathscr{L}^{>} := \{g \in \mathscr{L} \mid g > 0\}$.

A generalised theory of desirability

For the case of almost desirability, we consider $\mathscr{L}^{\geq} := \{g \in \mathscr{L} \mid g \geq 0\}$ also assume that we can equip the space with the order topology.

- Consider the underlying language $\mathscr{L}$ is given by a ordered vector space, with null element 0 and (order) unit 1
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What we are going to saying in this part holds also in the case discussed by Gert De Cooman on Thursday

A generalised theory of desirability

For the case of almost desirability, we also assume that $\mathscr{L}^\star \subseteq \mathscr{L}^\geq$ is a convex cone and thus that **Cl(posi(**$\mathscr{L}^\star$**))** $= \mathscr{L}^\star$.

- Consider the underlying language $\mathscr{L}$ is given by a ordered vector space, with null element 0 and (order) unit 1
- The space of gambles that should be "objectively" accepted is thus $\mathscr{L}^> := \{g \in \mathscr{L} \mid g > 0\}$.
- Now, assume that, for some reason, assessing in *general* that some thing is objectively acceptable is difficult, but we dispose of a series of criteria for which assessing if some thing is acceptable in the sense of belonging to $\mathscr{L}^\star \subseteq \mathscr{L}^>$ is doable.
  - We also assume that it is always the case that $1 \in \mathscr{L}^\star$
  - And also assume that $\mathscr{L}^\star$ is a convex cone (hence **posi(**$\mathscr{L}^\star$**)** $= \mathscr{L}^\star$) and thus that the convex $\mathscr{L}^\star \cup \{0\}$ is pointed at 0.

A generalised theory of desirability

- *Definition*: Consider a convex cone $\mathscr{L}^\star \subseteq \mathscr{L}^>$ as described before, and its induced partial order $f \leq^\star g$ iff $(g - f) \in \mathscr{L}^\star$. We say that a closure operator **cl** over $(\wp(\mathscr{L}), \subseteq)$ is compatible with $\mathscr{L}^\star$ if and only if
  - **respect assessable tautologies**: $\mathrm{cl}(\emptyset) = \mathscr{L}^\star$
  - **satisfies $\leq^\star$-dominance** : if $f \in \mathrm{cl}(A)$, and $f \leq^\star g$, then $g \in \mathrm{cl}(A)$

For the case of almost desirability, we first ask that C is definable by -1, and then that **cl** also satisfies the closure rule.

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  Thus, we say that a belief structure $(\wp(\mathscr{L}), \subseteq, \mathrm{cl}, C)$ in which is **cl** satisfies the two conditions above is a **generalised theory of quasi-desirability** if, in addition, C is definable by 0.

  We finally say that a generalised theory of quasi-desirability is **a generalised theory of desirability** if it is a strong belief structure in which the operator **cl** satisfies PS and ADD (i.e. $\mathrm{posi}(\mathrm{cl}(A)) = \mathrm{cl}(A)$, for every $A \subseteq \mathscr{L}$)

To better appreciate this remark, let's have a look at the probabilistic semantics for the standard theories of desirability, and the corresponding completeness theorems.

A generalised theor

- *Definition*: Consider a convex cone $\mathscr{L}^\star \subseteq \mathscr{L}^>$ as described before, and its induced partial order $f \leq^\star g$ iff $(g - f) \in \mathscr{L}^\star$. We say that a closure operator **cl** over $(\wp(\mathscr{L}), \subseteq)$ is compatible with $\mathscr{L}^\star$ if and only if
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Here one would like to be able to use a representation theorem for the maximal consistent theories so to link the structure to some other known context.

Probabilistic completeness(es)

Semantics for desirability

- Here the crux of the matter is to obtain the completeness result we are looking for by generating a probabilistic semantics via a representation theorem of hemispaces through duality (polarity) for the concerned vector space.

Such a move would also hold for the calculus of the TADG applied to quantum theory, more later.

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Probabilistic semantics

- A state is a linear functional L over the space of gambles $\mathscr{L}(\Omega)$ preserving the unit, i.e. $L(1) = 1$
- It corresponds to expectation with respect to a charge μ

$$L(g) = E_{\mu}(g) := (\mu \cdot g)$$

- Whenever L is positive, μ is a probability charge (in our cases a probability mass function and we identify it with a positive n-dimensional vectors of norm one)
  - We will from now on identify states with the corresponding charges and thus sometimes call also a state the latter

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Probabilistic semantics

- We say that a state μ is a *model* of a gamble $g \in \mathscr{L}(\Omega)$ if $E_\mu(g) \geq 0$ and write $\mu \vDash g$;
- It is a model of a set of gambles $\Gamma \subseteq \mathscr{L}(\Omega)$ if it is a model of each of its members, and write $\mu \vDash \Gamma$.

The beauty of being polar

A polarity theory for sets of desirable gambles

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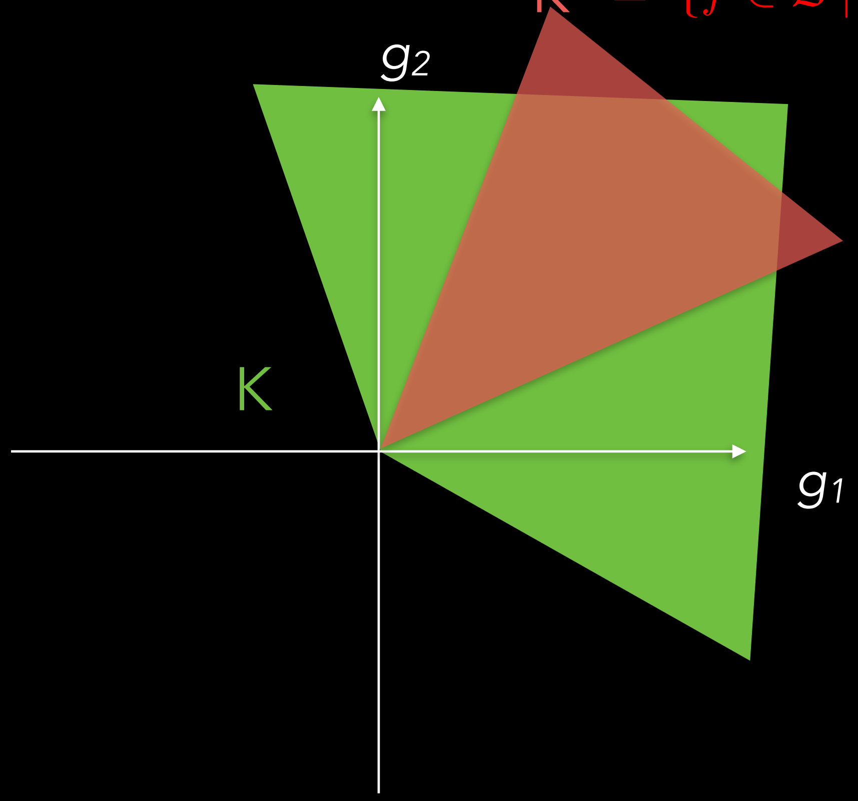
Abstract

Coherent sets of almost desirable gambles and credal sets are known to be equivalent models. That is, there exists a bijection between the two collections of sets preserving the usual operations, e.g. conditioning. Such a correspondence is based on the polarity theory for closed convex cones. Learning from this simple observation, in this paper we introduce a new (lexicographic) polarity theory for general convex cones and then we apply it in order to establish an analogous correspondence between coherent sets of desirable gambles and convex sets of lexicographic probabilities.
Keywords: Desirability; Credal sets; Lexicographic probabilities; Separation theorem; Polarity.

Polarity for TADG

$K^\bullet = \{f \in \mathcal{L} \mid (f \cdot g) \geq 0, \forall g \in K\}$



Polarity for TADG

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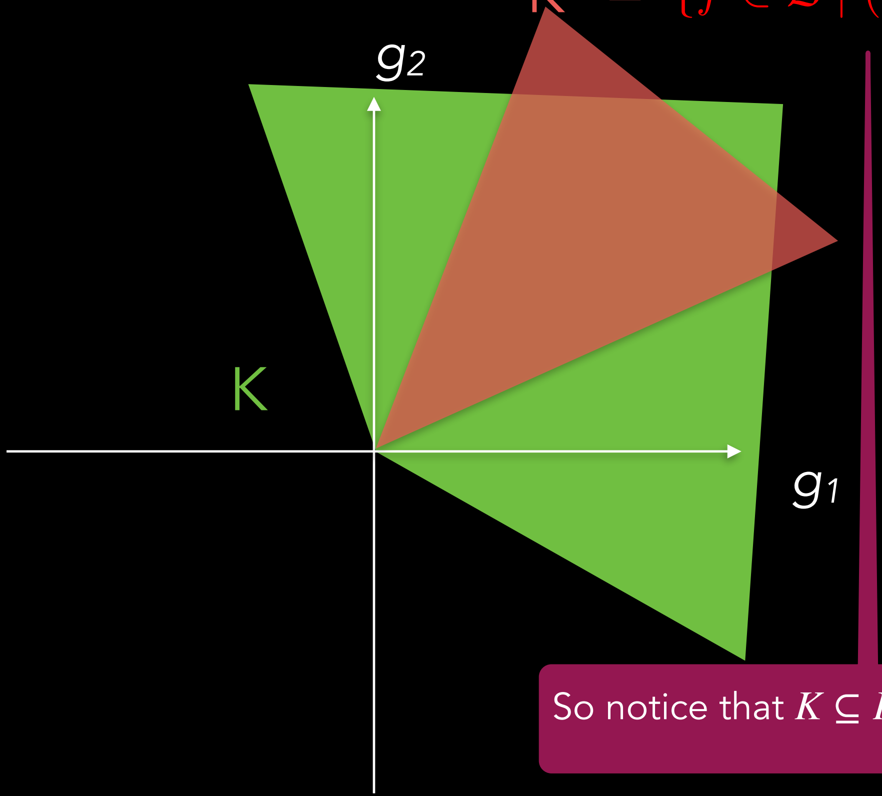
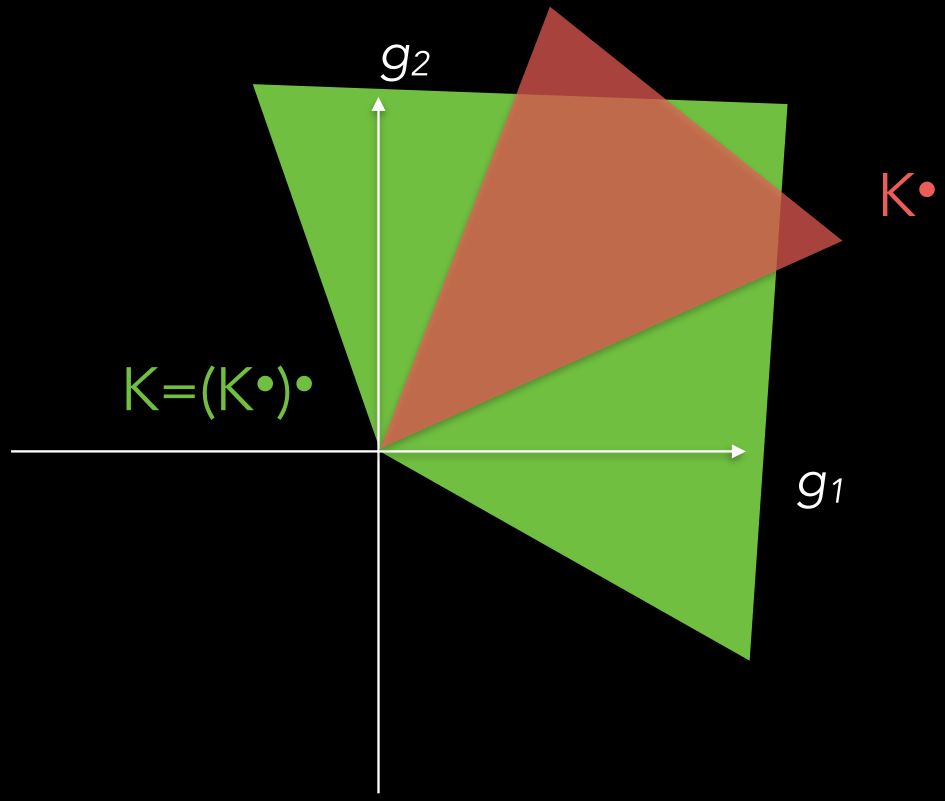
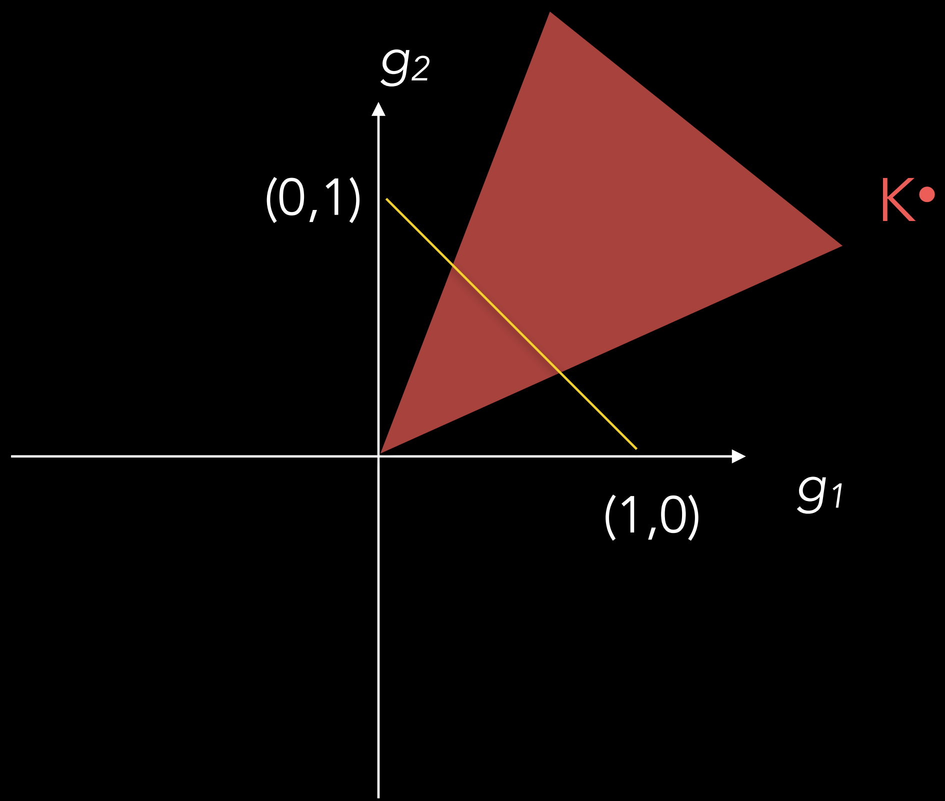


So notice that $K \subseteq K'$ implies $K^\bullet \supseteq (K')^\bullet$

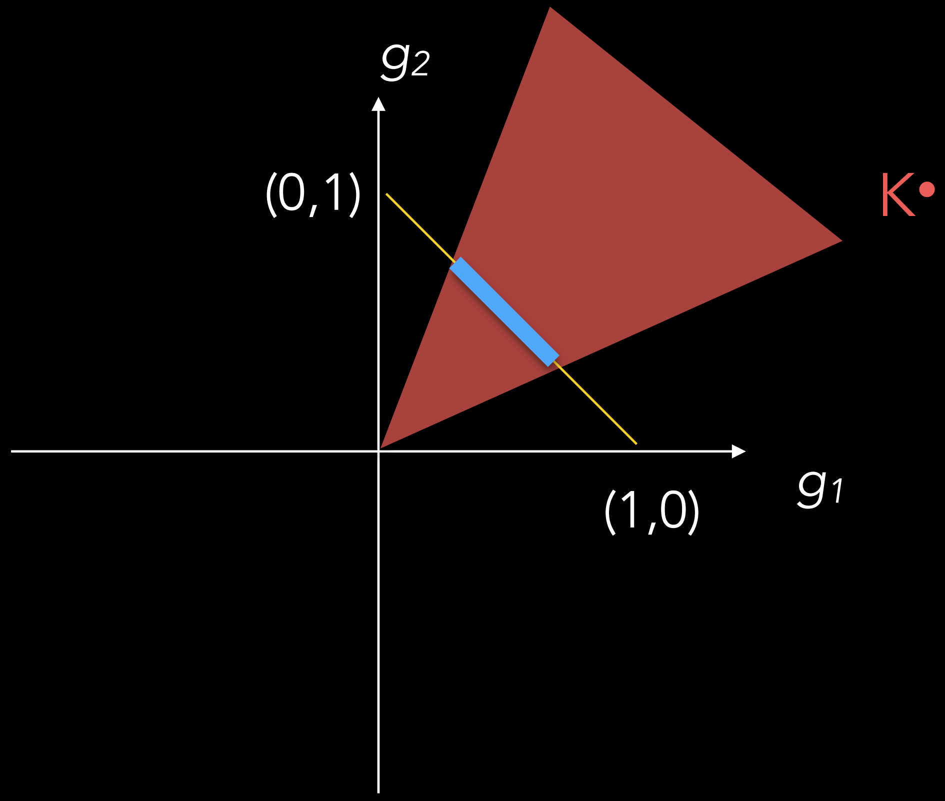
Polarity for TADG



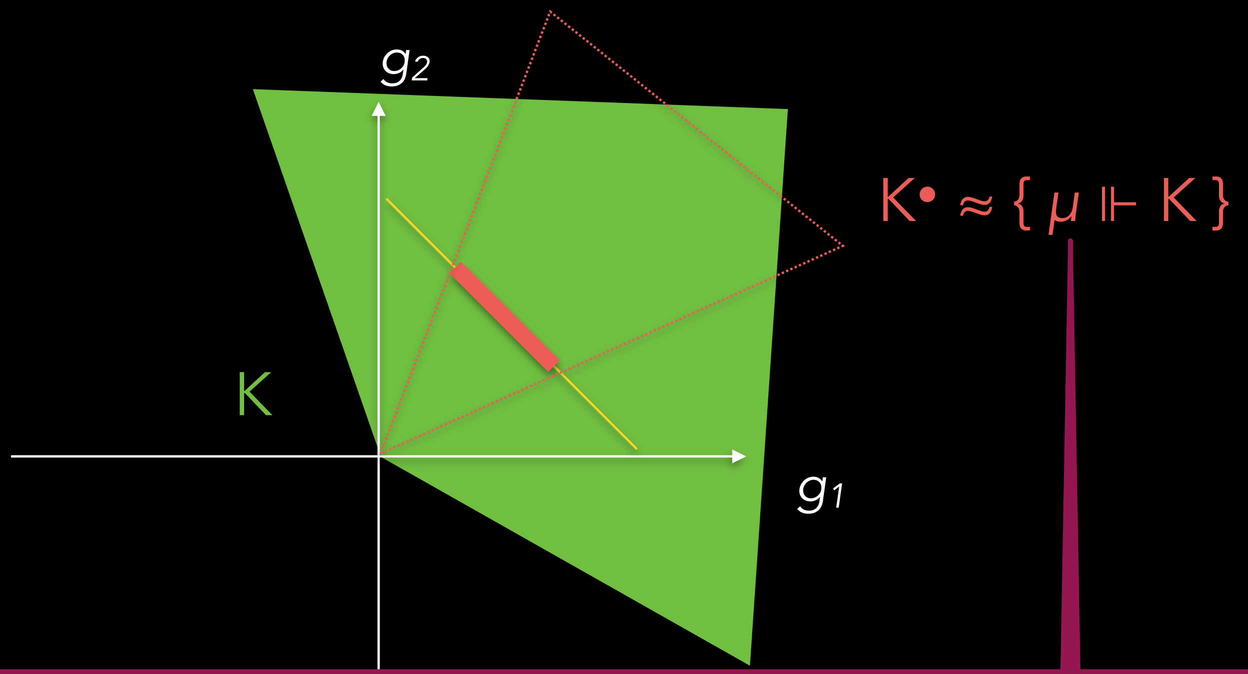
Polarity for TADG



Polarity for TADG



Polarity for TADG



We can identify the polar cone with the closed convex set of its section intersecting the collection of vectors corresponding to pmf, that is a credal set

Polarity for TADG

- Let $\mathbb{P}(\mathscr{L})$ be the collection of pmf in $\mathscr{L}$, and $\mathbf{R}(\mathscr{L}) \subseteq \wp(\mathbb{P}(\mathscr{L}))$ be the collection of all credal sets over $\mathscr{L}$.
- Consider the function $\mathscr{C} : \mathbf{C}_a(\mathscr{L}) \to \mathbf{R}(\mathscr{L})$ which maps a coherent set of almost desirable gambles $\mathscr{K} \in \mathbf{C}_a(\mathscr{L})$ into credal sets, that is

$$\mathscr{C}(\mathscr{K}) := \mathscr{K}^{\bullet} \cap \mathbb{P}(\mathscr{L})$$

Polarity for TADG

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$$\mathscr{C}(\mathscr{K}) := \mathscr{K}^\bullet \cap \mathbb{P}(\mathscr{L})$$

- *Fact*: The map $\mathscr{C} : \mathbf{C}_a(\mathscr{L}) \to \mathbf{R}(\mathscr{L})$ is a bijection, whose inverse is simply $\mathscr{C}^{-1}(C) = C^\bullet$, with C a credal set

Whenever $\mathscr{K} \in \mathbf{M}_a(\mathscr{L})$, we have that $\mathscr{C}(\mathscr{K})$ is a singleton, meaning that the following holds for every $\mathscr{K} \subseteq \mathscr{L}$

$$\mathscr{K} \in \mathbf{M}_a(\mathscr{L}) \text{ iff } \exists ! \mu : K = \{g \in \mathscr{L} \mid E_\mu(g) \geq 0\}$$

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Polarity for TADG

- When considering structures given by sets of gambles and a projection operation of conditioning, and similar with sets of pmf, it actually turns out that $\mathscr{C} : \mathbf{C}_a(\mathscr{L}) \to \mathbf{R}(\mathscr{L})$ is a isomorphism:

Theorem 6 *Let* $\mathcal{K} \in \mathbb{A}_n$ *and* $\Pi \subset \Omega$. *The following statements hold:*

(i) $(\mathcal{K}\rfloor_\Pi) \in \mathbb{A}_m$ *if and only if* $(\mathbf{C}(\mathcal{K})\rfloor_\Pi) \in \mathbb{C}_m$.

(ii) *If* $(\mathcal{K}\rfloor_\Pi) \in \mathbb{A}_m$, *then* $\mathbf{C}(\mathcal{K}\rfloor_\Pi) = (\mathbf{C}(\mathcal{K})\rfloor_\Pi)$.

Completeness for TADG

- *Theorem*: For every sequent $\Gamma \rhd g$, $\Gamma \vdash_{\mathfrak{A}} \varphi$ iff for every $\mu \in \mathbb{P}(\mathscr{L})$, $\mu \vDash \Gamma$ implies $\mu \vDash \varphi$

- *Proof*: Consider a sequent $\Gamma \rhd g$. Then we have that

$$\Gamma \vdash_{\mathfrak{A}} \varphi$$

iff

$$\mathsf{Cn}_a(\varphi) \subseteq \mathsf{Cn}_a(\Gamma)$$

iff

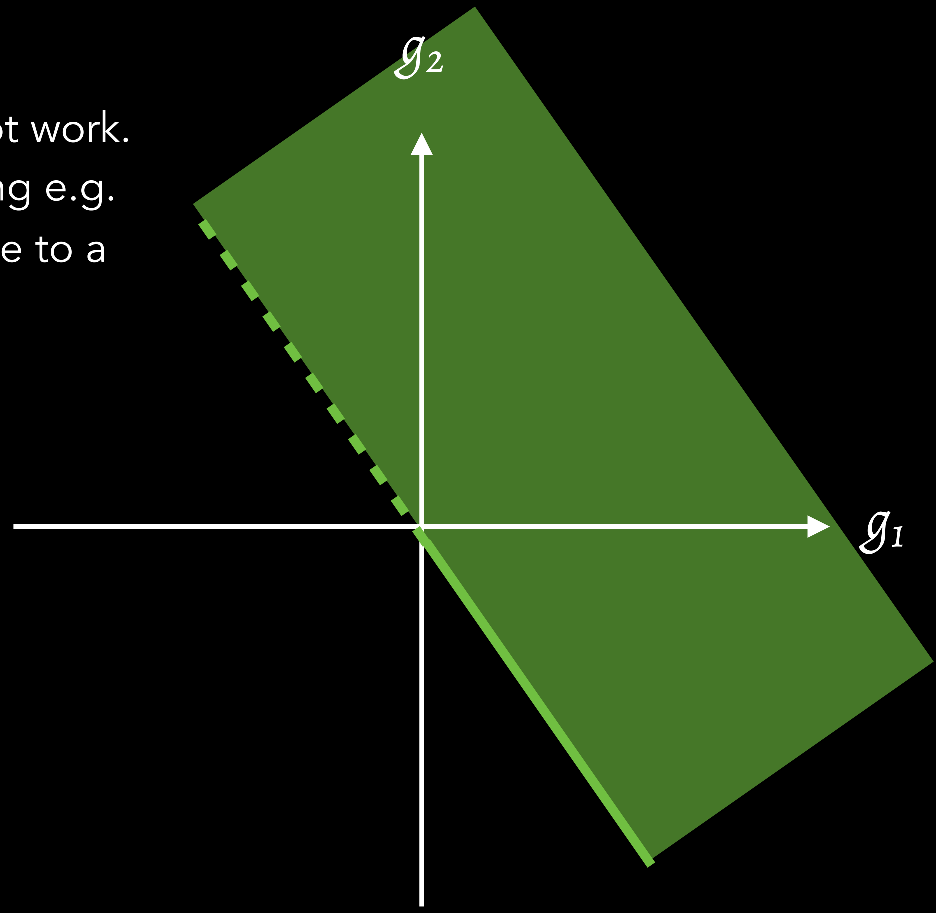
$$\mathbf{C}_a(\mathsf{Cn}(\varphi)) \supseteq \mathbf{C}_a(\mathsf{Cn}(\Gamma))$$

iff

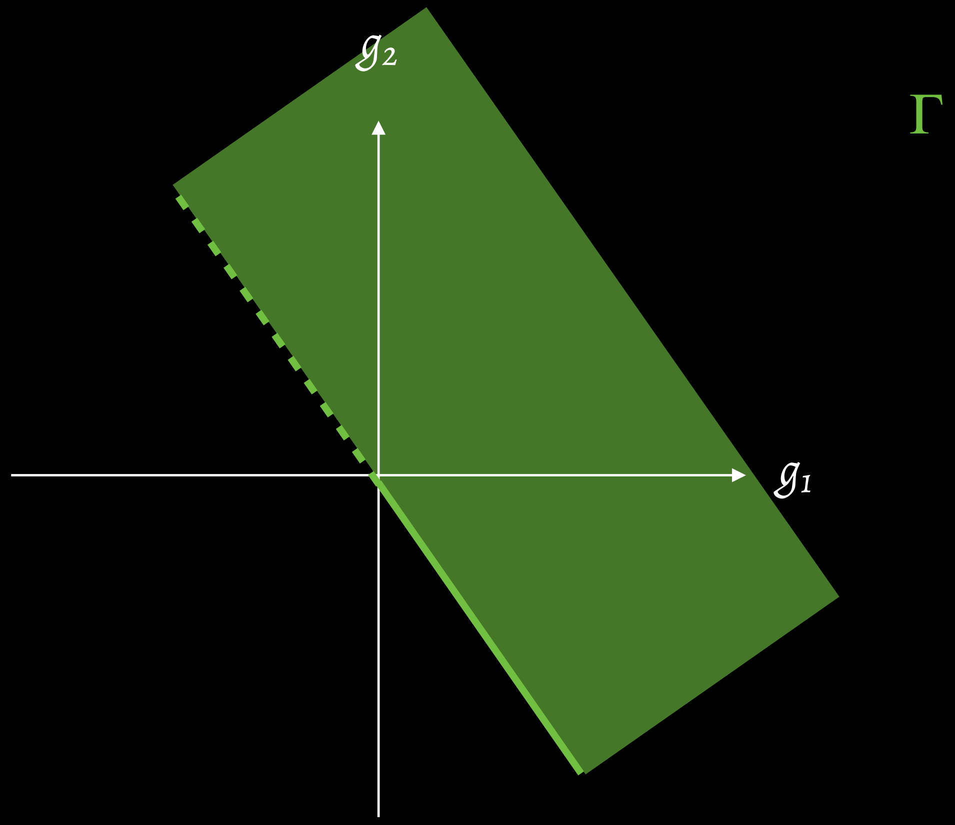
for every $\mu \in \mathbb{P}(\mathscr{L})$, $\mu \vDash \Gamma$ implies $\mu \vDash \varphi$

Polarity for TDG

- *Remark*: obviously polarity for TADG does not work. But even “relaxing” some conditions by taking e.g. as truth condition $E_\mu(g) > 0$ does not lead use to a complete probabilistic semantics.



Lexicographic duality



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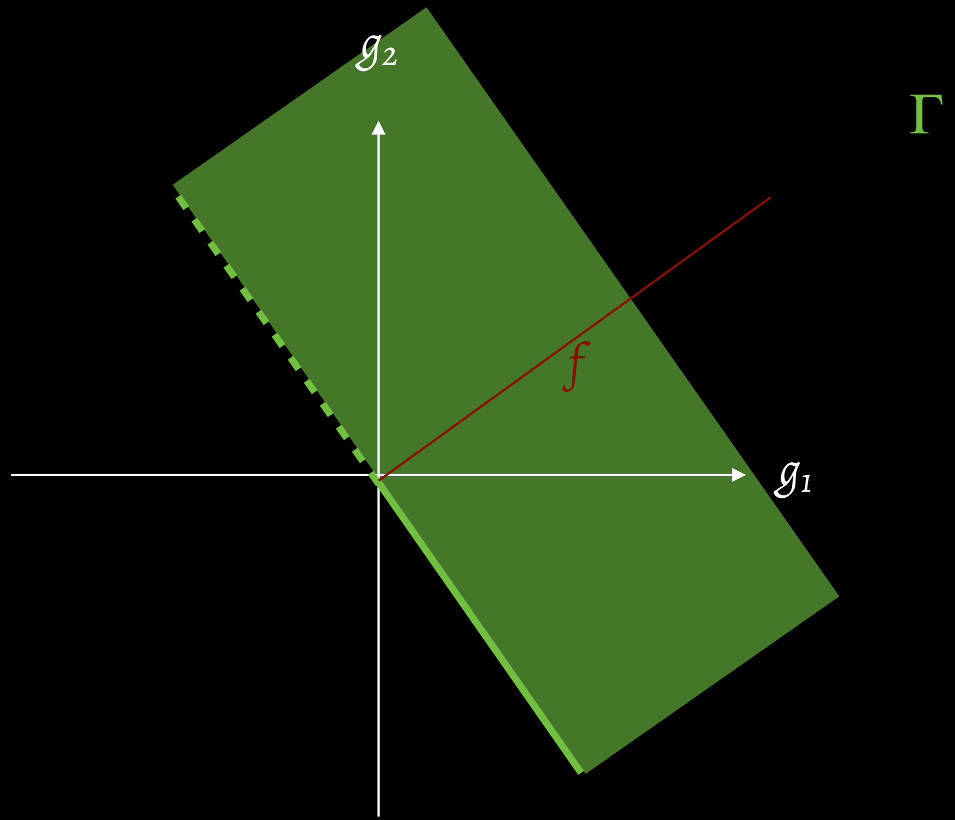

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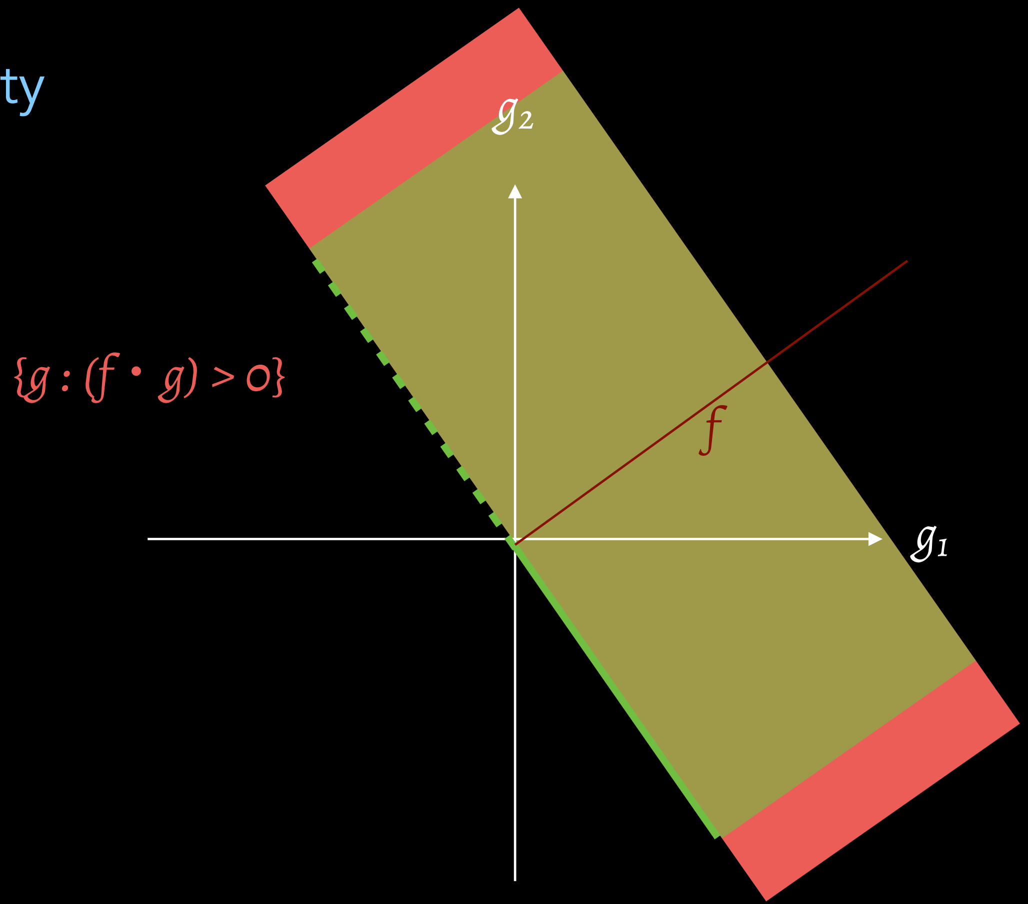

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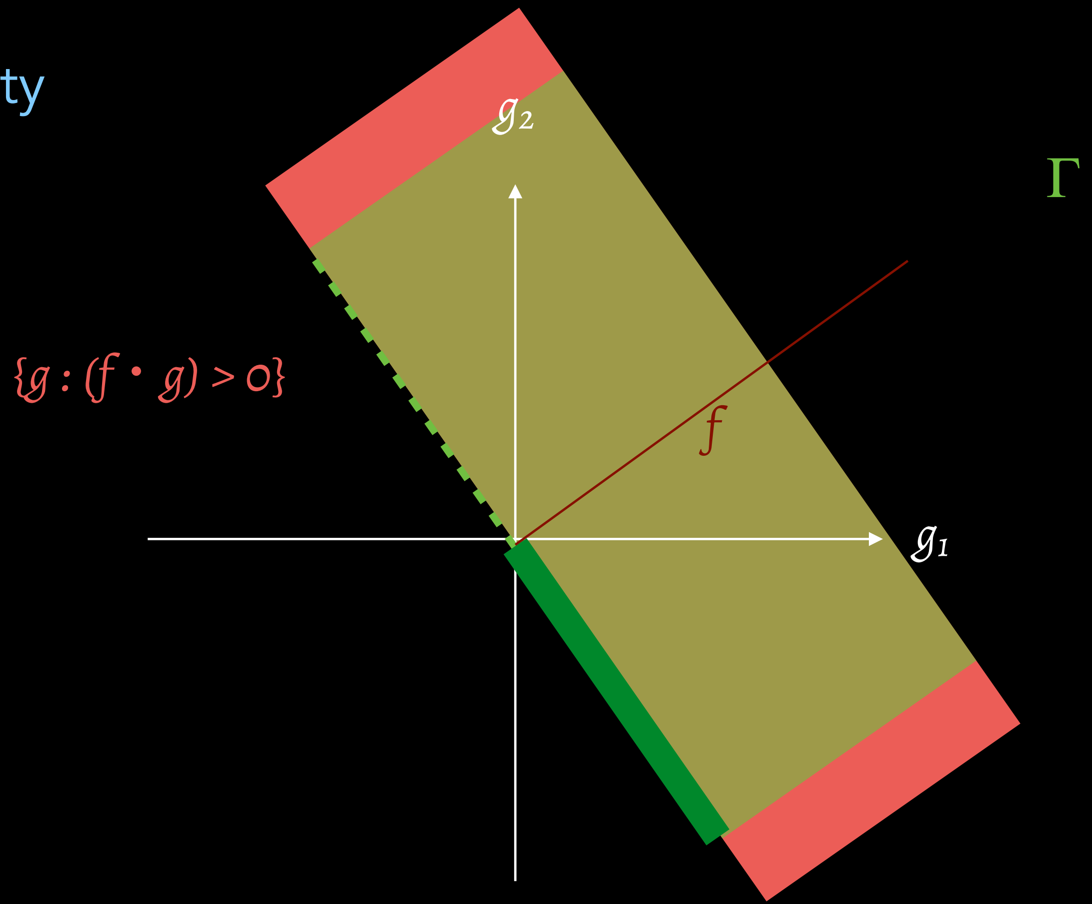
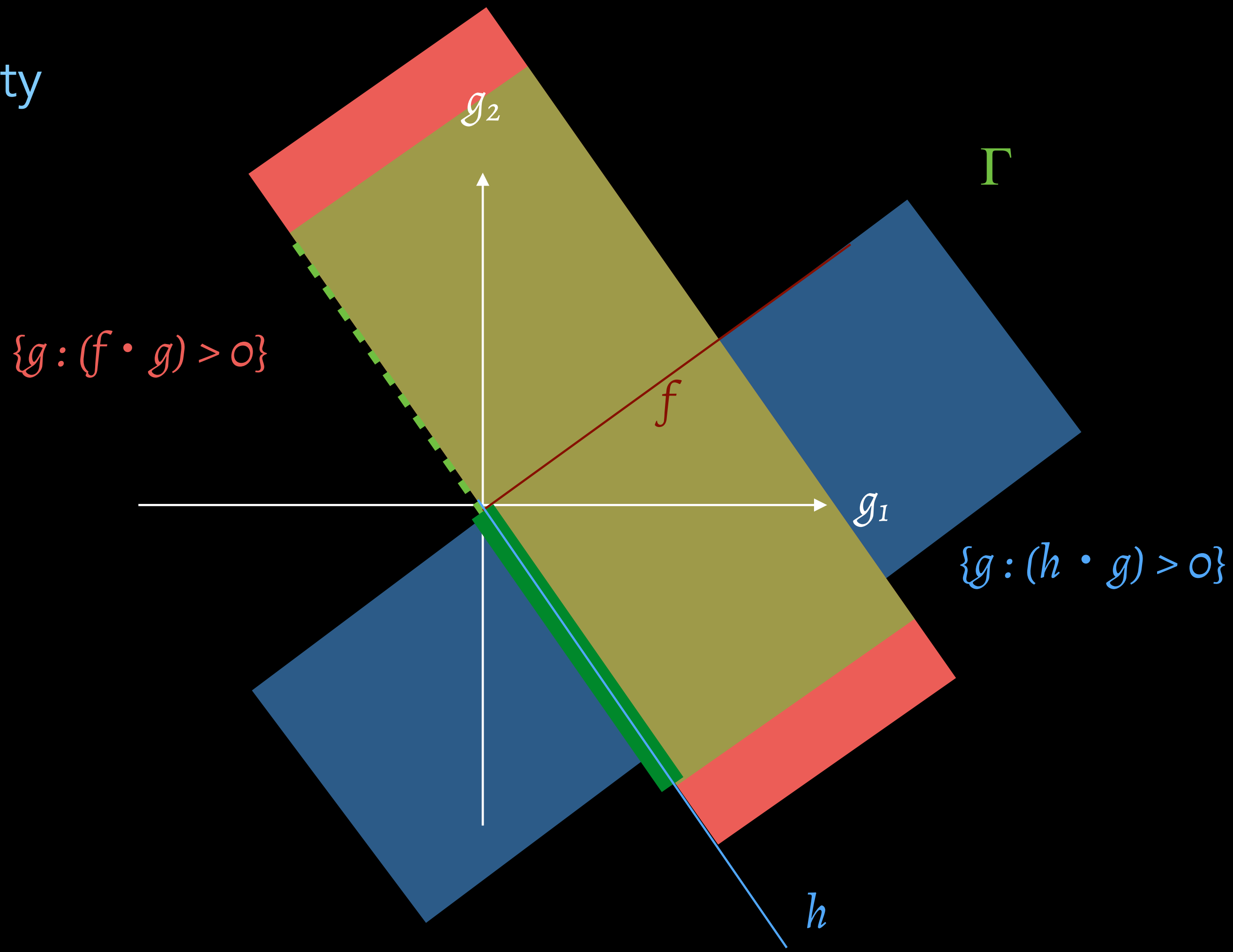
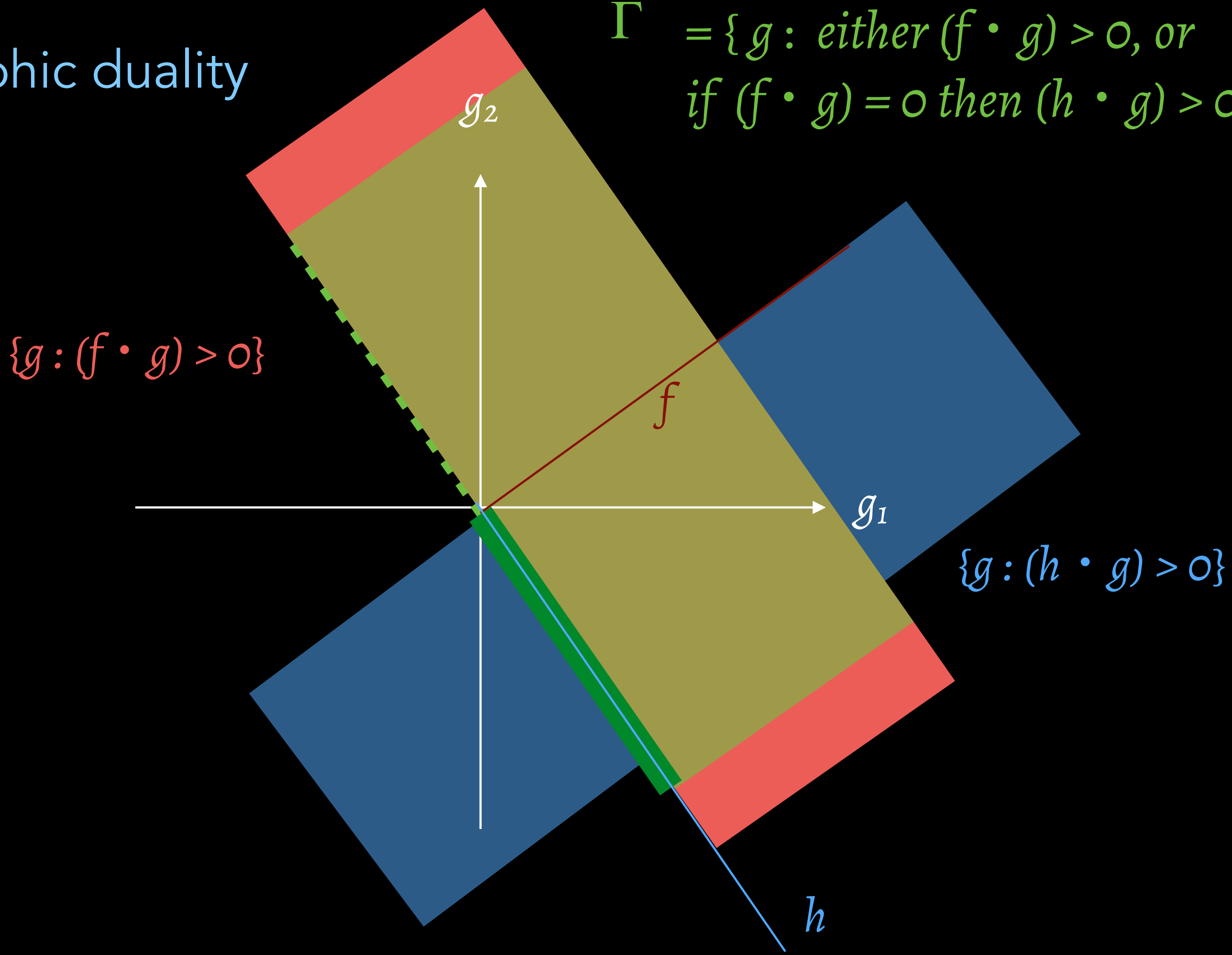


Lexicographic duality



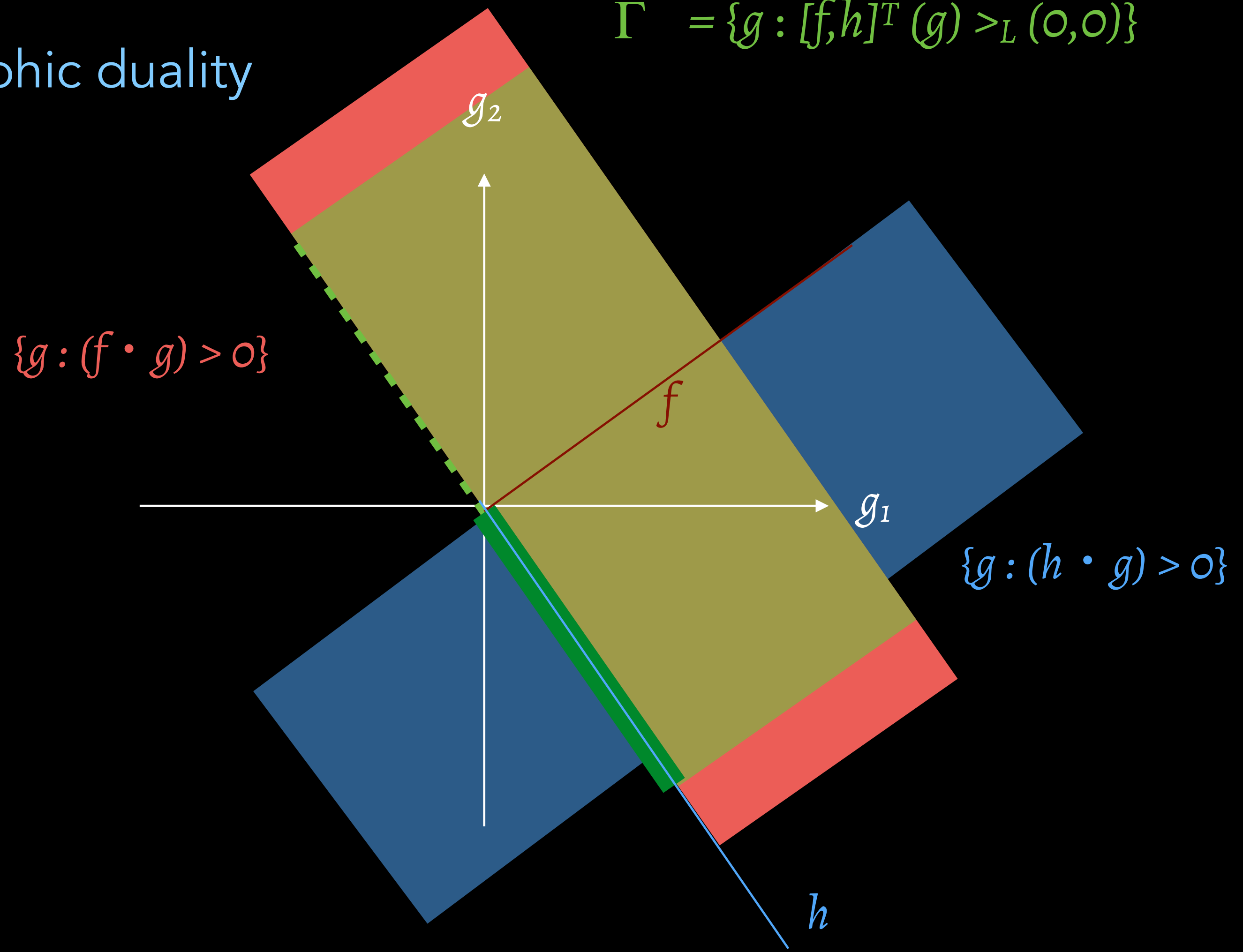
Lexicographic duality

$\Gamma = \{g : \text{ either } (f \cdot g) > 0, \text{ or if } (f \cdot g) = 0 \text{ then } (h \cdot g) > 0\}$

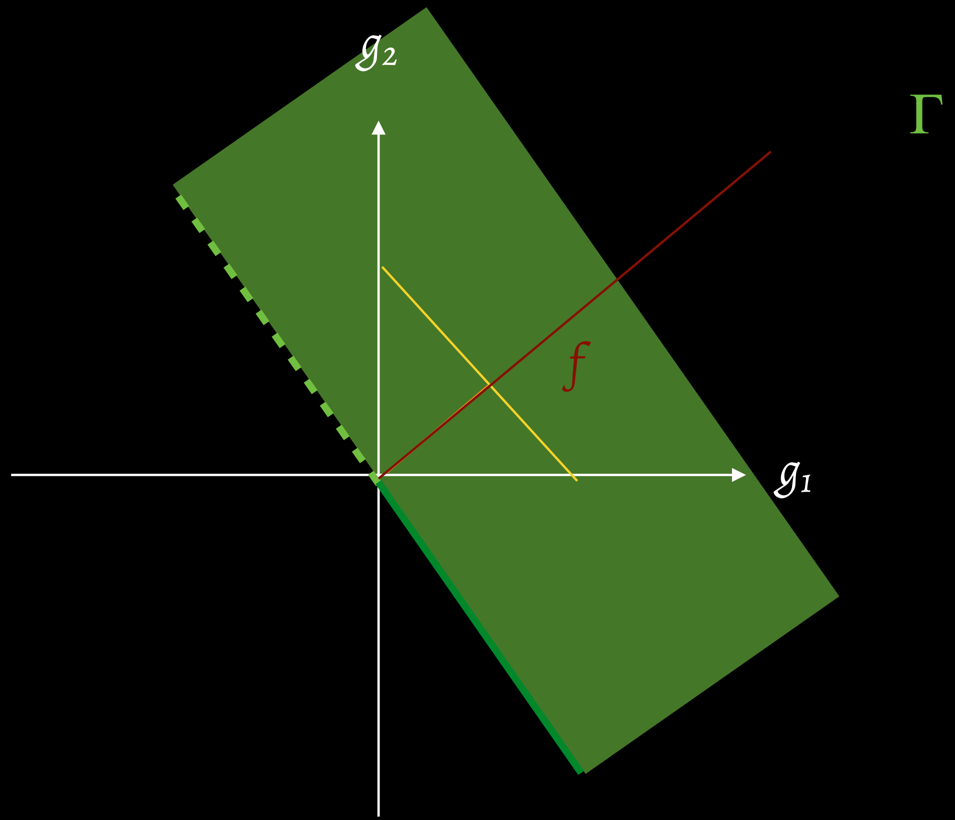


Lexicographic duality

$\Gamma = \{g : [f,h]^T (g) >_L (0,0)\}$



Lexicographic duality



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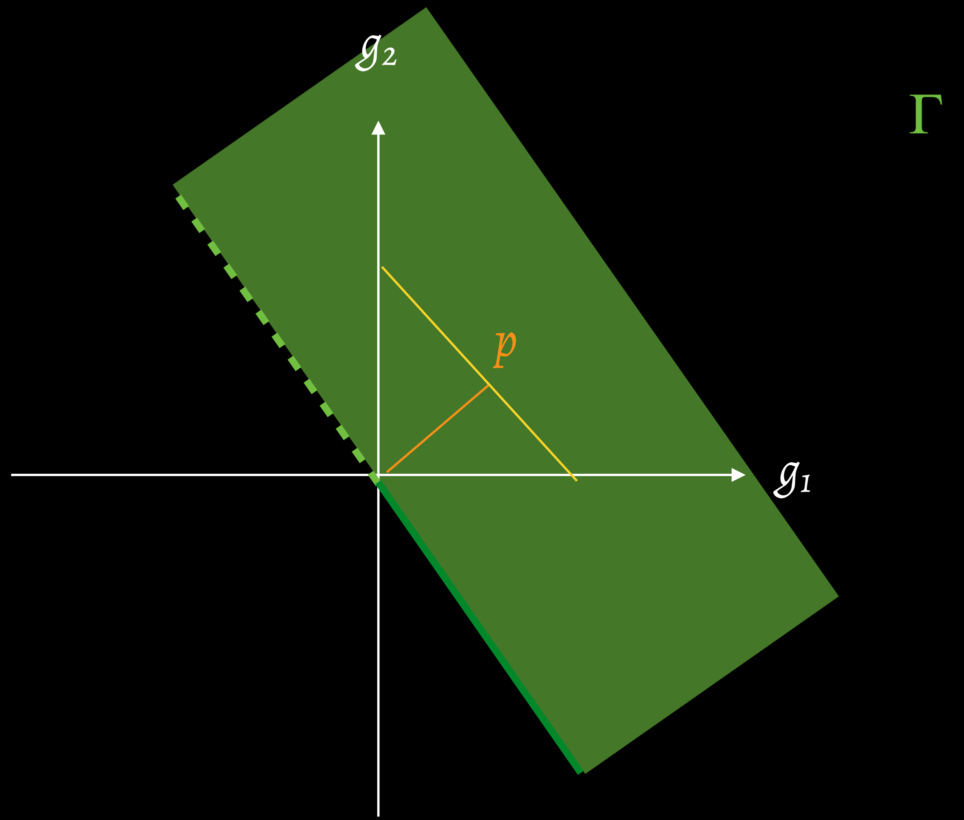

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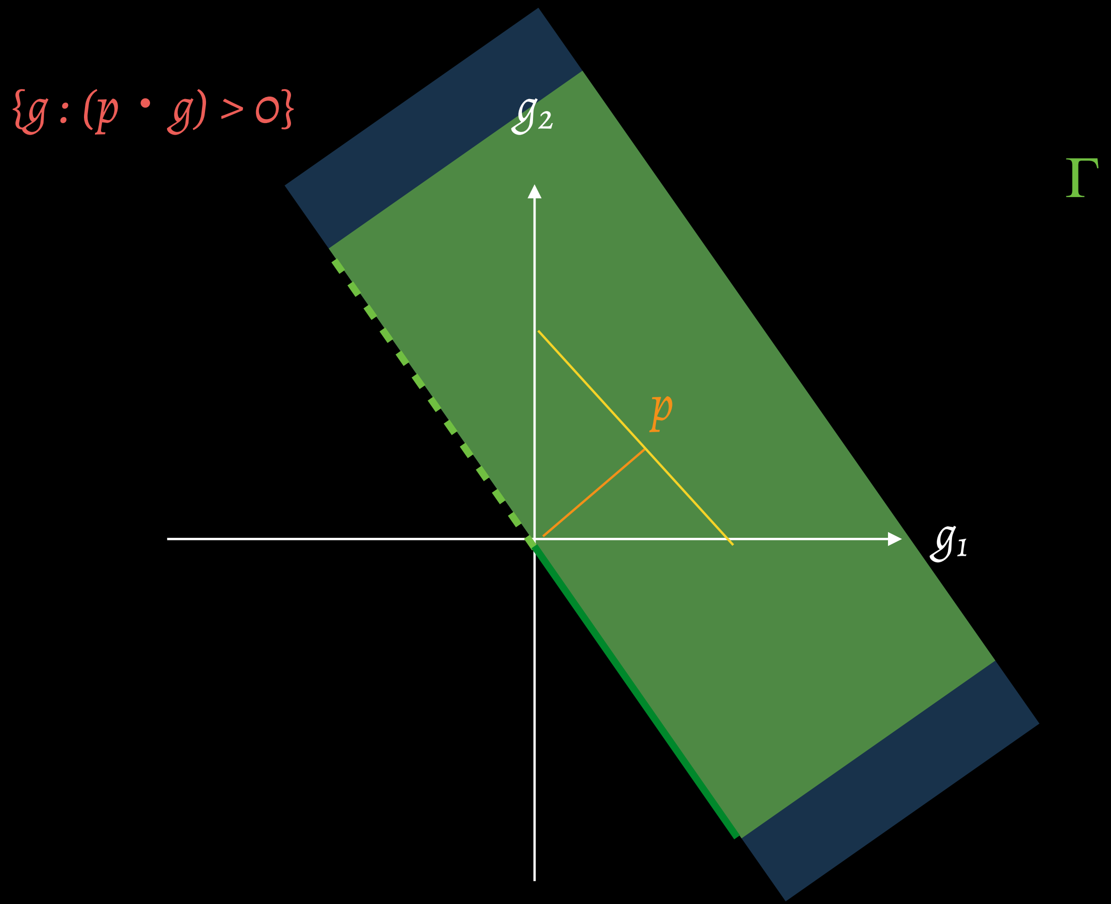

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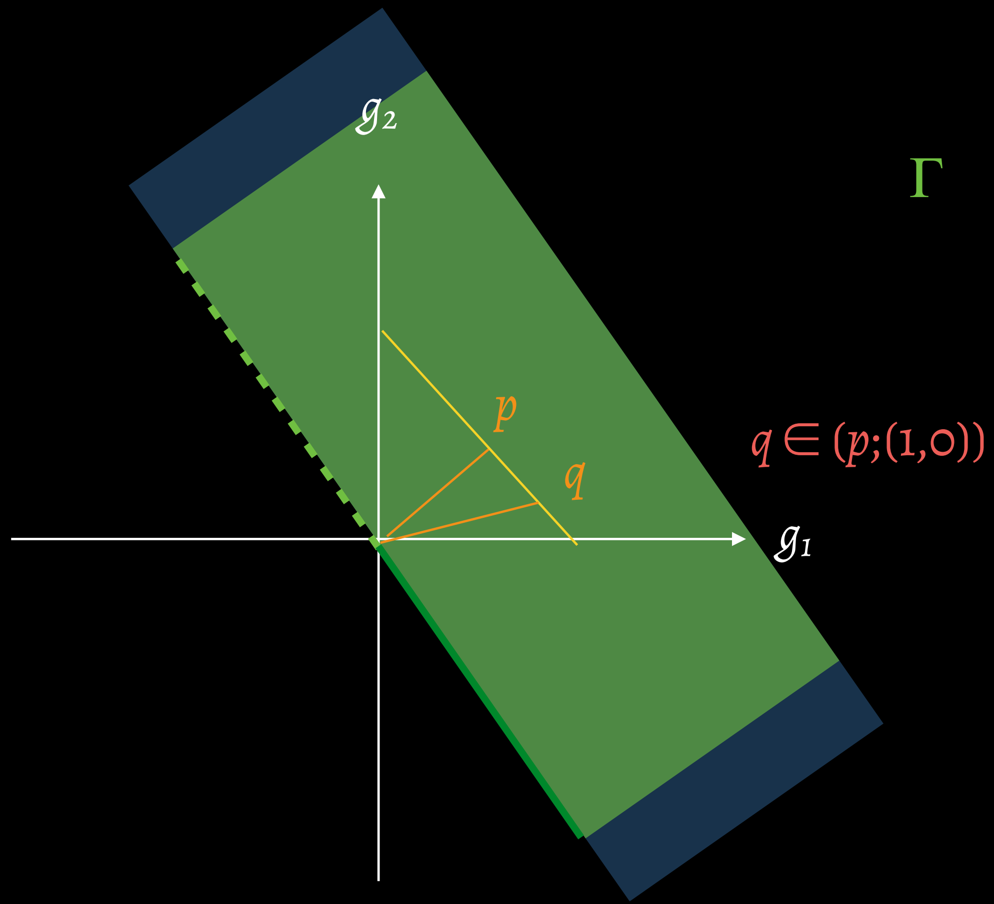
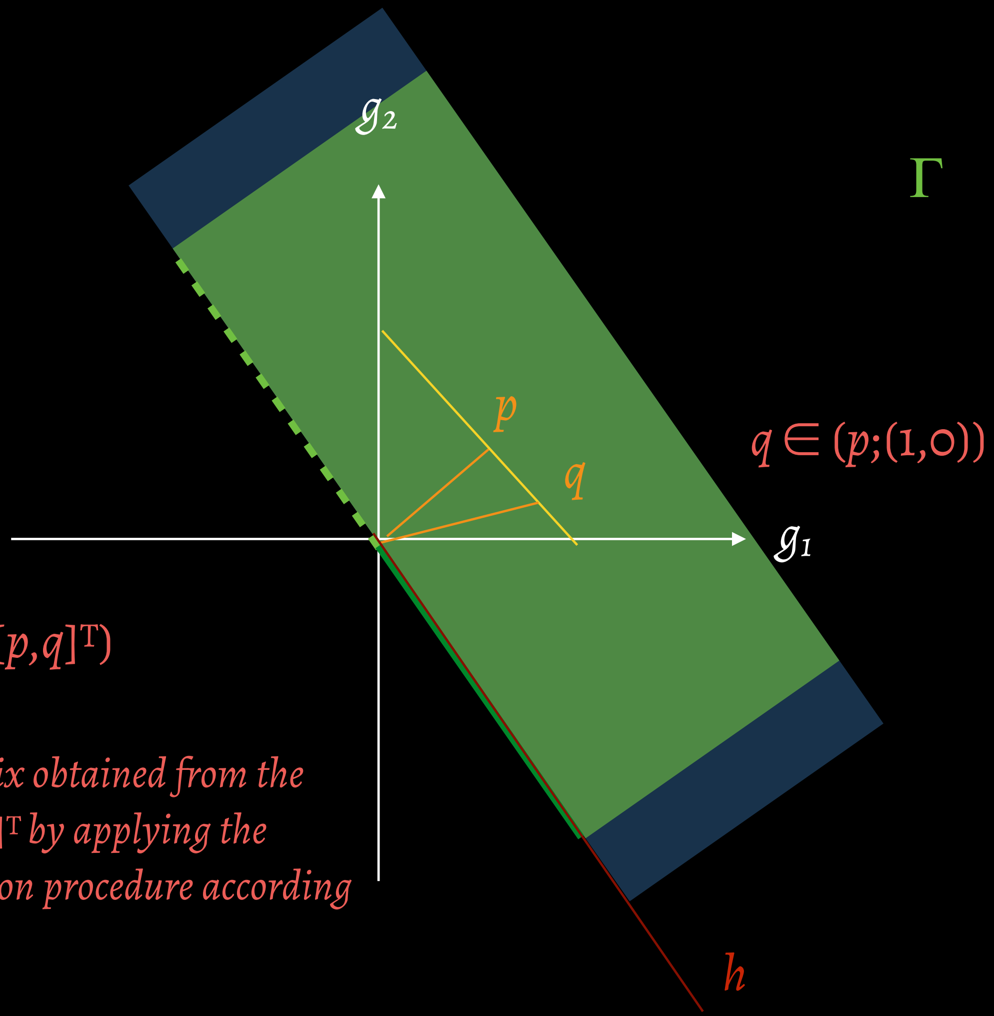


Lexicographic duality



$[p,h]^T = GS([p,q]^T)$

I.e. $[p,h]^T$ is the orthogonal matrix obtained from the full-rank stochastic matrix $[p,q]^T$ by applying the Gram–Schmidt orthogonalisation procedure according to the row order.

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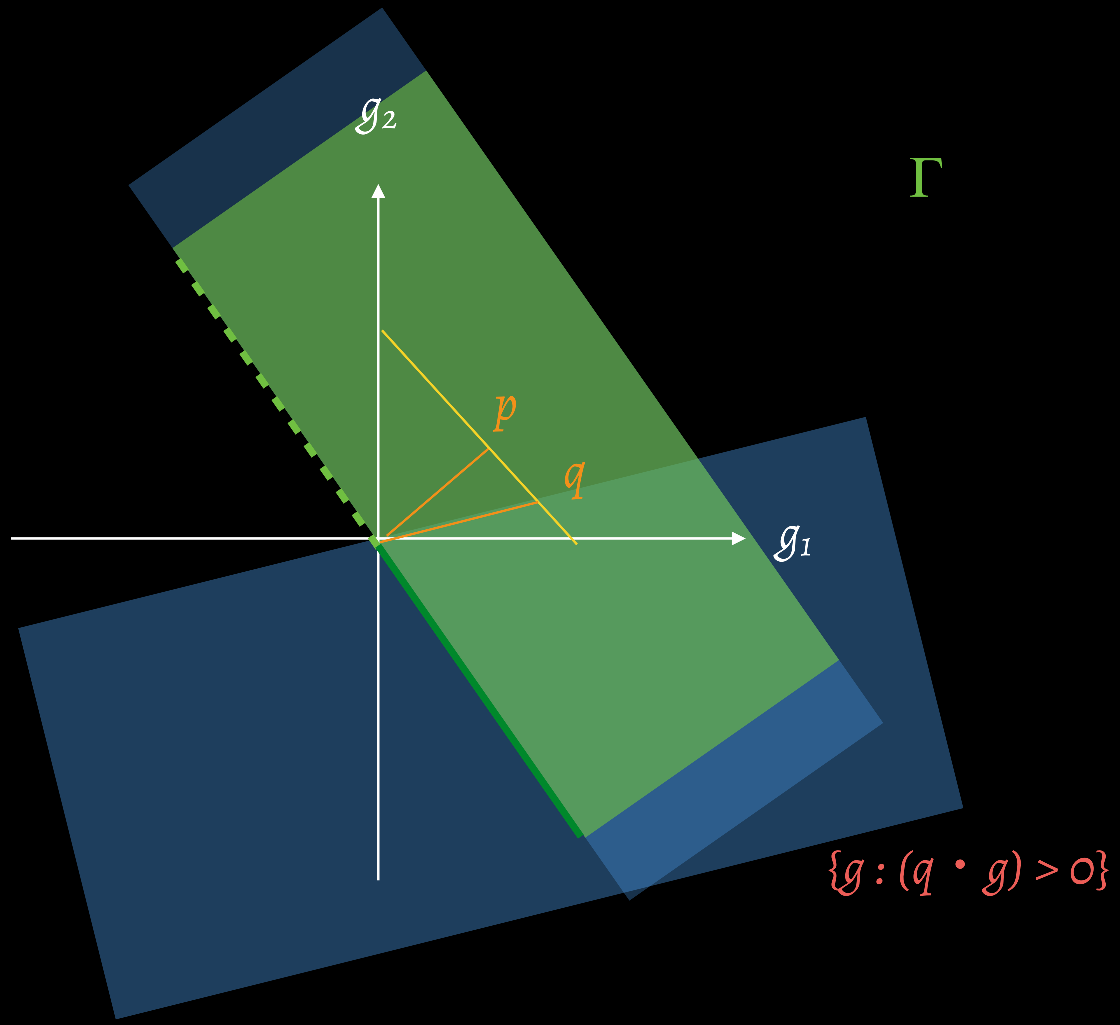
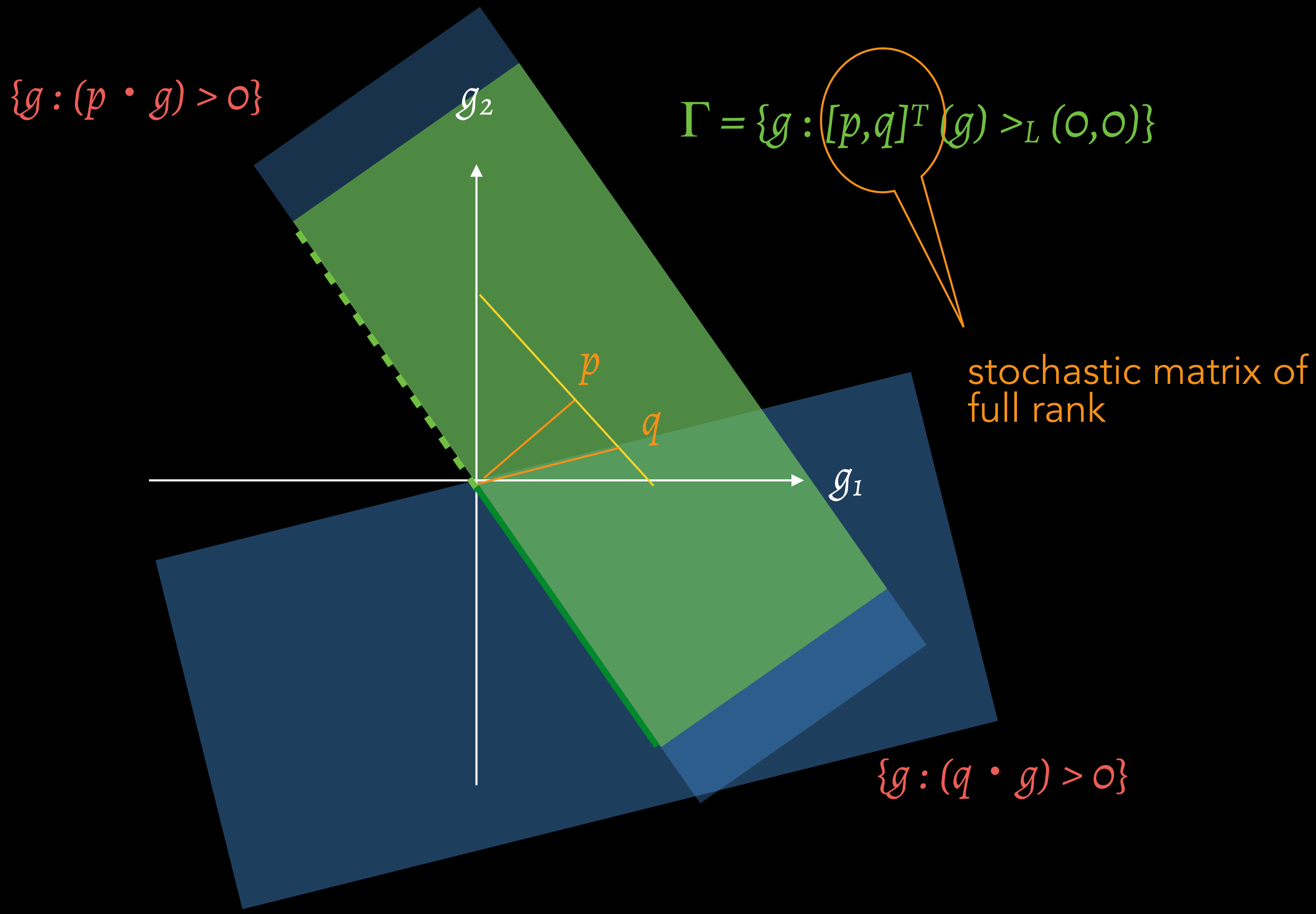
Lexicographic duality



Lexicographic duality



Probabilistic semantics (II)

Theorem: The structure of coherent sets of desirable gambles over Ω and the structure of "(lexicographic)-convex" sets of n-square stochastic matrices of full rank are isomorphic via lexicographic duality (polarity):

- $M^{\blacktriangledown} := \{g \in \mathscr{L} \mid P(g) >_L 0, \forall P \in M\}$, for $M \subseteq \mathbb{S}(\mathscr{L})$
- $K^{\vartriangle} := \{P \in \mathbb{S}(\mathscr{L}) \mid P(g) >_L 0, \forall g \in K\}$

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Let $P' \in \mathbb{M}_{n,m}$ be the matrix obtained by projecting on Π the conditioning $p(\cdot|\Pi)$, or taking 0_m when it is undefined, for each row p of $P \in \mathbb{T}_{n,n}$. Define $P\rfloor_\Pi$ as the matrix obtained from P' by applying rule (R). By an immediate application of properties of minors and cofactors, we get that $P\rfloor_\Pi \in \mathbb{T}_{m,m}$. Moreover $(P\rfloor_\Pi)\rfloor_\Delta = (P\rfloor_\Delta)$, for $\Delta \subset \Pi$. Hence, the following operation is always defined.

Definition 19 *Let $\mathcal{P} \subset \mathbb{T}_{n,n}$, with $n > 1$. Its conditioning on Π is the set $(\mathcal{P}\rfloor_\Pi) := \{(P\rfloor_\Pi) \mid P \in \mathcal{P}\} \subset \mathbb{T}_{m,m}$.*

From Definition 5, it is immediate to verify that $(\mathcal{K}\rfloor_\Pi) \in \mathbb{D}_m$ whenever $\mathcal{K} \in \mathbb{D}_n$, and that $\mathbb{D}_n$ is closed under conditioning. Moreover, $(\mathcal{K}\rfloor_\Pi) \in \mathsf{Max}(\mathbb{D}_m)$ whenever $\mathcal{K} \in \mathsf{Max}(\mathbb{D}_n)$. To conclude, we verify that polarity preserves conditioning.

Theorem 20 *Let $\mathcal{K} \in \mathbb{D}_n$, then $(\mathbf{G}(\mathcal{K})\rfloor_\Pi) = \mathbf{G}(\mathcal{K}\rfloor_\Pi) \in \mathbb{G}_m$.*

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Probabilistic semantics (II)

A lexicographic probability over Ω is a sequence $(p_1, \ldots, p_n)$ of probabilities over Ω.
Hence it can be seen as a stochastic matrix $P := [p_1, \ldots, p_n]^T$.

A n-square stochastic matrix P of full rank is a *model* of a a gamble *g* if

$$P(g) = [E_{p1}(g), \ldots, E_{pk}(g)]^T >_L 0,$$

and write $P \Vdash g$.
It is a model of a set Γ if it is a model of each of its members, and write $P \Vdash \Gamma$.

We denote by $\mathbb{S}$ the collection of stochastic matrices of full rank (the fixed dimension and the space are implicit)

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Completeness for TDG via lexicographic duality

- *Theorem*: For every sequent $\Gamma \rhd g$, $\Gamma \vdash_{\mathfrak{D}} \varphi$ iff for every $P \in \mathbb{S}(\mathscr{L})$, $P \vDash \Gamma$ implies $P \vDash \varphi$

- *Proof*: Consider a sequent $\Gamma \rhd g$. Then we have that

$$\Gamma \vdash_{\mathfrak{D}} \varphi$$
iff
$$\mathrm{Cn}(\varphi) \subseteq \mathrm{Cn}(\Gamma)$$
iff
$$(\mathrm{Cn}(\varphi))^{\vartriangle} \supseteq (\mathrm{Cn}(\Gamma))^{\vartriangle}$$
iff
for every $P \in \mathbb{S}(\mathscr{L})$, $P \vDash \Gamma$ implies $P \vDash \varphi$

Part II: Extending the logic of desirability

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Varieties of negation

Subminimal negation

- Let $(A, \vdash)$ be some consequence system, and consider a unary function $\neg : A \times A \to A$
- A minimal structural property for negation as unary operator, is that if two "things" are inter-derivable (modulo a given set of assessments), hence their "negation" too is inter-derivable, that is
  - for every $a, b \in A$, and every $\Gamma \subseteq A$
    - If $\Gamma, a \vdash b$ and $\Gamma, b \vdash a$, then $\Gamma, \neg a \vdash \neg b$ ($\neg$-functionality)
  - a unary function $\neg : A \times A \to A$ that satisfies functionality is called a **subminimal negation**

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First made explicit in the works by Kosta Dosen (1999) and especially, in a systematic way, by Almudena Colacito, Dick De Jongh, & Ana Lucia Vargas (2016).

From now on, when not derivable, we always assume this property

Soft Comput (2017) 21:165–174
DOI 10.1007/s00500-016-2391-8

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Subminimal negation

Almudena Colacito[1] · Dick de Jongh[1] · Ana Lucia Vargas[1]

Minimal and intuitionistic negation

- Let $(A, \vdash)$ be some consequence system, and consider a subminimal negation $\neg : A \times A \to A$
- We may then have rules for introducing and eliminating a "negation"
  - for every $a, b \in A$, and every $\Gamma \subseteq A$
    - If $\Gamma, a \vdash b$ and $\Gamma, a \vdash \neg b$, then $\Gamma \vdash \neg a$ (¬-introduction)
    - If $\Gamma \vdash a$ and $\Gamma \vdash \neg a$, then $\Gamma \vdash b$ (¬-elimination / ex-contradictio sequitur quod libet)
  - a subminimal negation $\neg : A \times A \to A$ that satisfies ¬-introduction is called a **minimal negation**
  - a minimal negation $\neg : A \times A \to A$ that satisfies ¬-elimination is called a **intuitionistic negation**

Minimal and intuitionistic ne

Functionality follows from the introduction rule. In fact assume $\Gamma, a \vdash b$ and $\Gamma, b \vdash a$. Now, by diluation $\Gamma, a, \neg b \vdash b$ and by reflexivity $\Gamma, a, \neg b \vdash \neg b$, hence by introduction $\Gamma, \neg b \vdash \neg a$.

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Proposition: A minimal negation is an antitonic operation, meaning that it satisfies the following **contraposition** law: for every $a, b \in A$, and every $\Gamma \subseteq A$, if $\Gamma, a \vdash b$, then $\Gamma, \neg b \vdash \neg a$.

Proof: Assume $\Gamma, a \vdash b$. By dilution $\Gamma, a, \neg b \vdash b$. By reflexivity $\Gamma, a, \neg b \vdash \neg b$. Thus by ¬-introduction we conclude $\Gamma, \neg b \vdash \neg a$.

Minimal and intuitionistic ne

Contraposition does not imply ¬-introduction, hence one could actually consider the system given by functionality + contraposition, without the rule of introduction.

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Minimal and intuitionistic ne

Proposition: A minimal negation satisfies the law of introduction of the double negation: for every $a \in A$, and every $\Gamma \subseteq A$, $\Gamma, a \vdash \neg\neg a$

Proof: By reflexivity $\Gamma, a, \neg a \vdash a$ and $\Gamma, a, \neg a \vdash \neg a$. Thus by ¬-introduction we conclude $\Gamma, a \vdash \neg\neg a$.

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Negation and the falsum

- Definable belief structure naturally arises when the underlying language has an operator acting as a subminimal negation, and the following introduction rule holds, for every $b \in A$, and every $\Gamma \subseteq A$:
  - If $\Gamma \vdash b$ and $\Gamma \vdash \neg b$, then $\Gamma \vdash \bot$ (⊥-introduction)
- When in the presence of a minimal negation, one also typically ask for a weak rule of elimination, which is specular to ¬-introduction:
  - If $\Gamma, b \vdash \bot$ then $\Gamma \vdash \neg b$ (minimal ⊥-elimination)

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The idea of a *minimal* (or pre-)falsum is to have a specific symbol representing the fact of being able to derive a some thing and its negation (opposite).

- *Fact*: Given a consequence system $(A, \vdash)$ containing a minimal negation, and satisfying (⊥-introduction) and (minimal ⊥-elimination), one can check that, for every $\Gamma \subseteq A$, whenever $\Gamma \neq \{\bot\}$, it holds that

$$\bot \in \mathrm{Cn}_{\vdash}(\Gamma) \text{ iff } \{b, \neg b\} \subseteq \mathrm{Cn}_{\vdash}(\Gamma), \text{ for some } b \in A.$$

Classical negation

- To get classical negation, we need some additional property. First we state the following principle:
  - for every $a, b \in A$, and every $\Gamma \subseteq A$
    - If $\Gamma, \neg a \vdash a$, then $\Gamma \vdash a$ (Curry's law)
  - a intuitionistic negation $\neg : A \times A \to A$ that satisfies Curry's law is called a **classical negation.**

- *Theorem*: A classical negation satisfies the following properties:
  - If $\Gamma, \neg a \vdash b$ and $\Gamma, \neg a \vdash \neg b$, then $\Gamma \vdash a$ (Reduction ad absurdum)
  - $\Gamma, \neg\neg a \vdash a$ (Double negation elimination)

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Proposition: Every classical negation satisfies RAA and the law of double negation.

Proof: For the first claim, by applying $\neg$-elimination to $\Gamma, \neg a \vdash b$, and $\Gamma, \neg a \vdash \neg b$, we get $\Gamma, \neg a \vdash a$. We conclude by Curry's law. Finally, by applying RAA to facts $\neg\neg a, \neg a \vdash \neg a$ and $\neg\neg a, \neg a \vdash \neg\neg a$, we get the double negation elimination law.

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  - $\Gamma, \neg\neg a \vdash a$ (Double negation elimination)

Proposition: Every subminimal negation that satisfies RAA is classical.

Proof: $\neg$-introduction is immediate by dilution, and Curry's law by using reflexivity. For $\neg$-elimination we reason as follows. From the previous proposition, RAA implies double negation elimination. Thus, from the latter and cut, assuming $\Gamma, a \vdash b$ and $\Gamma, a \vdash \neg b$, we get $\Gamma, \neg\neg a \vdash b$ and $\Gamma, \neg\neg a \vdash \neg b$, and by RAA, we conclude that $\Gamma \vdash \neg a$.

Three main pure calculi of negation (w/ falsum)

Classical negation
Introduction, elimination and Curry’s law

↑

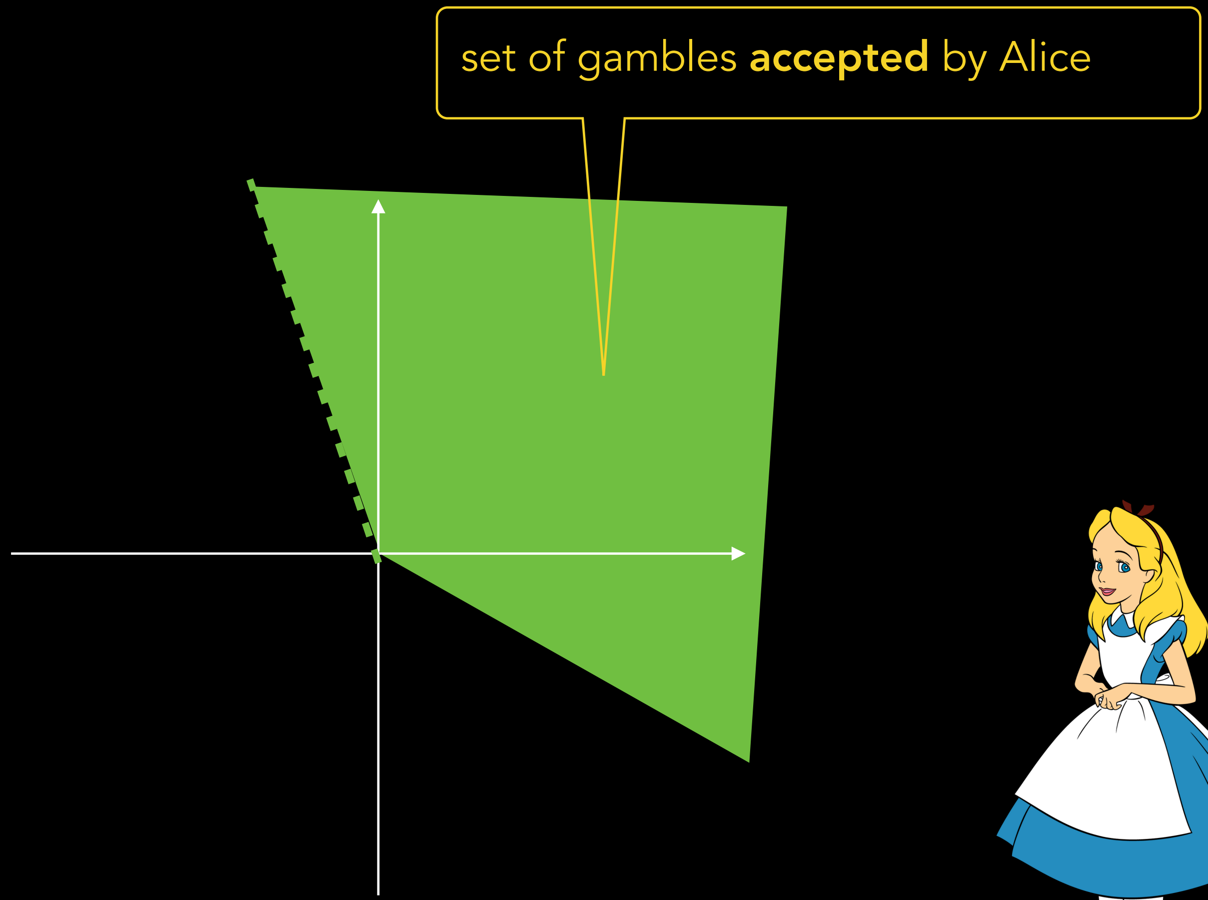
Intuitionistic negation
Introduction and elimination

↑

Minimal negation
Introduction (and weak elimination)

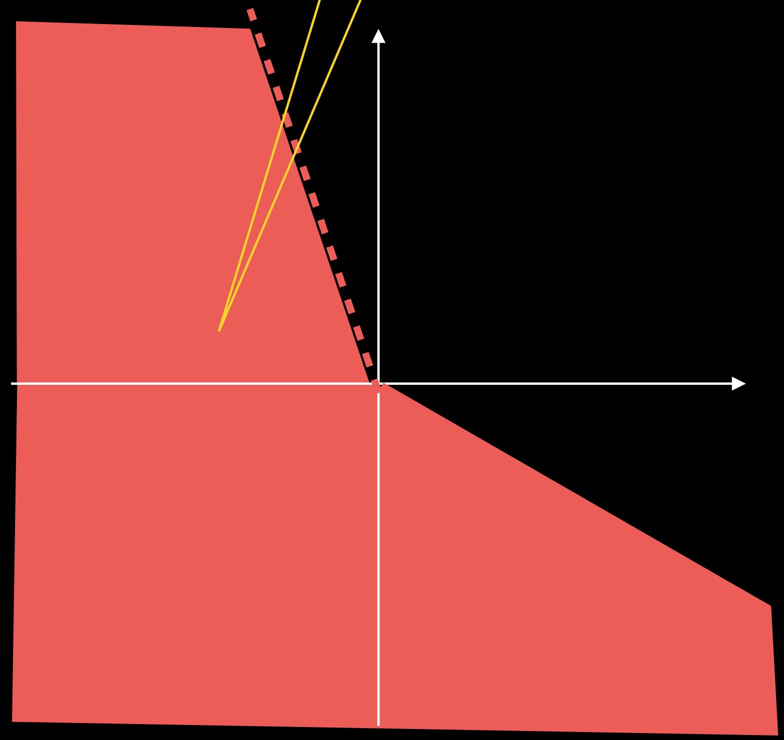
The logic of accept and reject

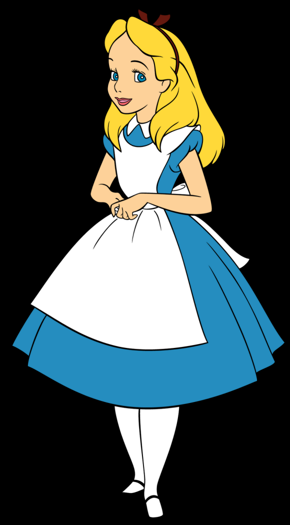
Rejecting gambles



Rejecting gambles

set of gambles that Alice **does not accept**





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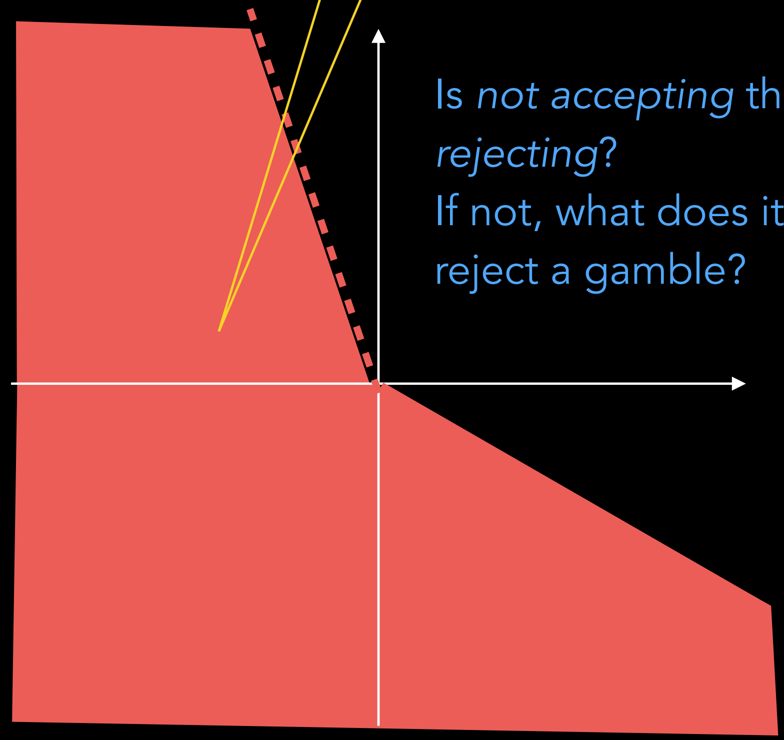
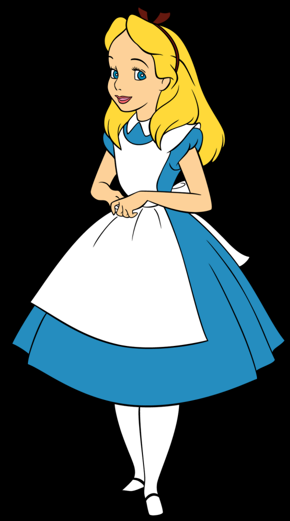


Is *not accepting* the same act as *rejecting*?
If not, what does it means for Alice to reject a gamble?



Rejecting gambles

In (Quaeghebeur, De Cooman & Hermans 2015), a general framework for modelling uncertainty, going beyond TDG, is presented based around the idea that gambles are categorised into accepted and rejected ones.

E.g. avoiding sure loss: Alice **rejects** all negative gambles


International Journal of Approximate Reasoning
Volume 57, February 2015, Pages 69-102

Accept & reject statement-based uncertainty models

Erik Quaeghebeur [a b 1], Gert de Cooman [a], Filip Hermans [a]

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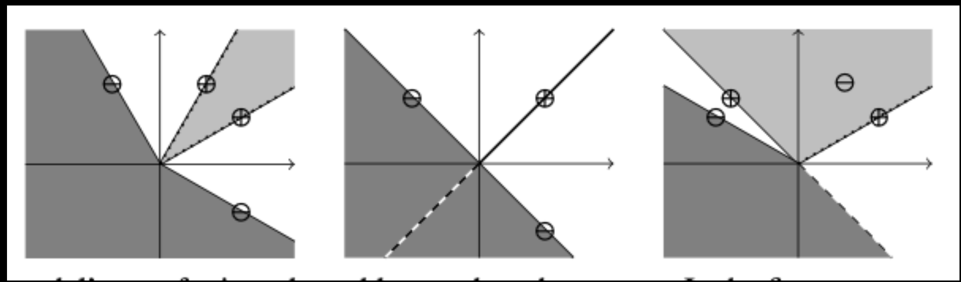
Highlights

- We develop a framework for modelling and reasoning based on a pair of gamble sets.



Rejecting gambles

- New principles (e.g. no confusion, no limbo), idea of extension for sets of accepted and rejected gambles.
- However, rejecting (a gamble) is not really treated as a “logical operation”, as with accepting (a gamble) in TDG

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If I accept X and Y, then I "rationally" have to accept X+Y.
But what if I accept X and reject Y?
*Hence, **what does it mean to reject a gamble, from a logical point of view?***

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*Hence, **what does it mean to reject a gamble, from a logical point of view?***

The logic of accept & reject

We follow (Bendall 1979); Humberstone 2000), and define a signed formulas as an expression of the form $[+]g$ or $[-]g$, with g a gamble.

- the expression $[+]g$ reads "g is accepted",
- the expression $[-]g$ reads "g is rejected".

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Notre Dame Journal of Formal Logic
Volume XX, Number 1, January 1979
NDJFAM

NEGATION AS A SIGN OF NEGATIVE JUDGMENT

KENT BENDALL

1 *Introduction* We need to form negative as well as affirmative statements because we need to mark falsity as well as truth, to register rejection as false as well as acceptance as true, and to deny as well as to assert. But we do not need an embeddable negation operator any more than we need an embeddable affirmation operator, provided operators are available for forming conjunctions, disjunctions, conditionals, and universal and existential generalizations. This thesis, which is examined and

LLOYD HUMBERSTONE

THE REVIVAL OF REJECTIVE NEGATION

First received on 13 August 1999; Final version received on 4 February 2000

ABSTRACT. Whether assent ("acceptance") and dissent ("rejection") are thought of as speech acts or as propositional attitudes, the leading idea of *rejectivism* is that a grasp of the distinction between them is prior to our understanding of negation as a sentence operator, this operator then being explicable as applying to *A* to yield something assent to which is tantamount to dissent from *A*. Widely thought to have been refuted by an argument of Frege's, rejectivism has undergone something of a revival in recent years, especially in writings by Huw Price and Timothy Smiley. While agreeing that Frege's argument does not refute the position, we shall air some philosophical qualms about it in Section 5, after a thorough examination of the formal issues in Sections 1–4. This discussion draws on – and seeks to draw attention to – some pertinent work of Kent Bendall in the 1970s.

KEY WORDS: negation, rejection, assertion, denial, rules, consequence relations, signed formulas, connectives.

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The logic of accept & reject

We follow (Bendall 1979); Humberstone 2000), and define a signed formulas as an expression of the form $[+]g$ or $[-]g$, with g a gamble.

- the expression $[+]g$ reads "g is accepted",
- the expression $[-]g$ reads "g is rejected".

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KEY WORDS: negation, rejection, assertion, denial, rules, consequence relations, signed formulas, connectives.

Idea: define a calculus where "formulas" are unidimensional from which to derive the characterisation results from (Quaeghebeur, De Cooman & Hermans 2015)

The logic of accept & reject

$$\frac{}{\Gamma \triangleright g} \quad g>0 \qquad \text{(APG)}$$

$$\frac{\Gamma \triangleright g}{\Gamma \triangleright \lambda g} \quad \lambda>0 \qquad \text{(PS)}$$

$$\frac{\Gamma \triangleright g \qquad \Gamma \triangleright f}{\Gamma \triangleright g+f} \qquad \text{(ADD)}$$

$$\frac{\Gamma \triangleright \bot}{\Gamma \triangleright g} \qquad (\bot\text{-elimination})$$

The logic of accept & reject

$$\frac{}{\Gamma_s \triangleright [+]g} \quad g>0 \qquad \text{(APG)}$$

$$\frac{\Gamma_s \triangleright [+]g}{\Gamma_s \triangleright [+]\lambda g} \quad \lambda>0 \qquad \text{(PS)}$$

$$\frac{\Gamma_s \triangleright [+]g \qquad \Gamma_s \triangleright [+]f}{\Gamma_s \triangleright [+](g+f)} \qquad \text{(ADD)}$$

$$\frac{\Gamma_s \triangleright [+]\bot}{\Gamma_s \triangleright [+]g} \qquad \text{(}\bot\text{-elimination)}$$

The logic of accept & reject

$$\frac{}{\Gamma_s \triangleright [*]g} \quad [*]g \in \Gamma_s \qquad \text{(R)}$$

$$\frac{\Phi_s \triangleright [*]g}{\Gamma_s \triangleright [*]g} \quad \Phi_s \subseteq \Gamma_s \qquad \text{(D)}$$

$$\frac{\Gamma_s \triangleright [*]g \qquad \Gamma_s, [*]g \triangleright [\diamond]f}{\Gamma_s \triangleright [\diamond]f} \qquad \text{(cut)}$$

where $*/\diamond = +,-$

The logic of accept & reject

The first is just functionality for negation but for rejection, and thus captures a minimal structural property for the corresponding unary operator

$$\frac{\Gamma_s, [+]g \triangleright [+]f \qquad \Gamma_s, [+]f \triangleright [+]g}{\Gamma_s, [-]g \triangleright [-]f} \quad \text{(N)}$$

“ given Γ_s, if the acceptance of g implies the acceptance of f, and vice versa, then rejecting g forces me to reject f too”

The logic of accept & reject

The next rules are the standard introduction and elimination rules (cf. with the negation operation)

(no limbo)

$$\frac{\Gamma_s, [+]g \triangleright [+]\bot}{\Gamma_s \triangleright [-]g} \quad ([-]\text{-I})$$

"given Γ_s, if accepting g leads to an incoherence, i.e. I am forced to accept the falsum, then I reject g"

(no confusion)

$$\frac{\Gamma_s \triangleright [+]g \qquad \Gamma_s \triangleright [-]g}{\Gamma_s \triangleright [+]\bot} \quad [-]\text{-E})$$

The logic of accept & reject

From no limbo, we get the following "standard" contraposition rule for rejection

(contraposition)

$$\frac{\Gamma_{s,}[+]g \triangleright [+]f}{\Gamma_s\,,\,[-]f \triangleright [-]g} \qquad \text{(CON)}$$

The reasoning is the same as for the case with standard negation. Use dilution, reflexivity and then apply no limbo.

The logic of accept & reject

The next pair of rules explicitly connects rejection with multiplication by -1 (“classical implicit internal negation”), and mimic somehow in this sense forms of contraposition rules.

(“**from -1 to [-]**”)

$$\frac{\Gamma_s, [+]g \rhd [+]\text{-}f}{\Gamma_s, [+]f \rhd [-]g} \quad \text{(W.Conn. 1)}$$

(“**from [-] to -1**”)

$$\frac{\Gamma_s, [+]g \rhd [-]f}{\Gamma_s, [+]f \rhd [+]\text{-}g} \quad \text{(W.Conn. 2)}$$

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The logic of accept & reject

The next rule are the standard defining properties of classical negation expressed in terms of multiplication by -1 and rejection:

$$\frac{\Gamma_{s,}[+]\text{-}g \rhd [+]\bot}{\Gamma_s \rhd [+]g} \qquad \text{(RAA)}$$

We could also have a variant of RAA without making explicit -1, but we should then have in the background some principle connecting -1 and rejection. Similar, variants of Curry's law, with explicit mention of -1 or not.

$$\frac{\Gamma_s \rhd [-]\text{-}g}{\Gamma_s \rhd [+]g} \qquad \text{(DN)}$$

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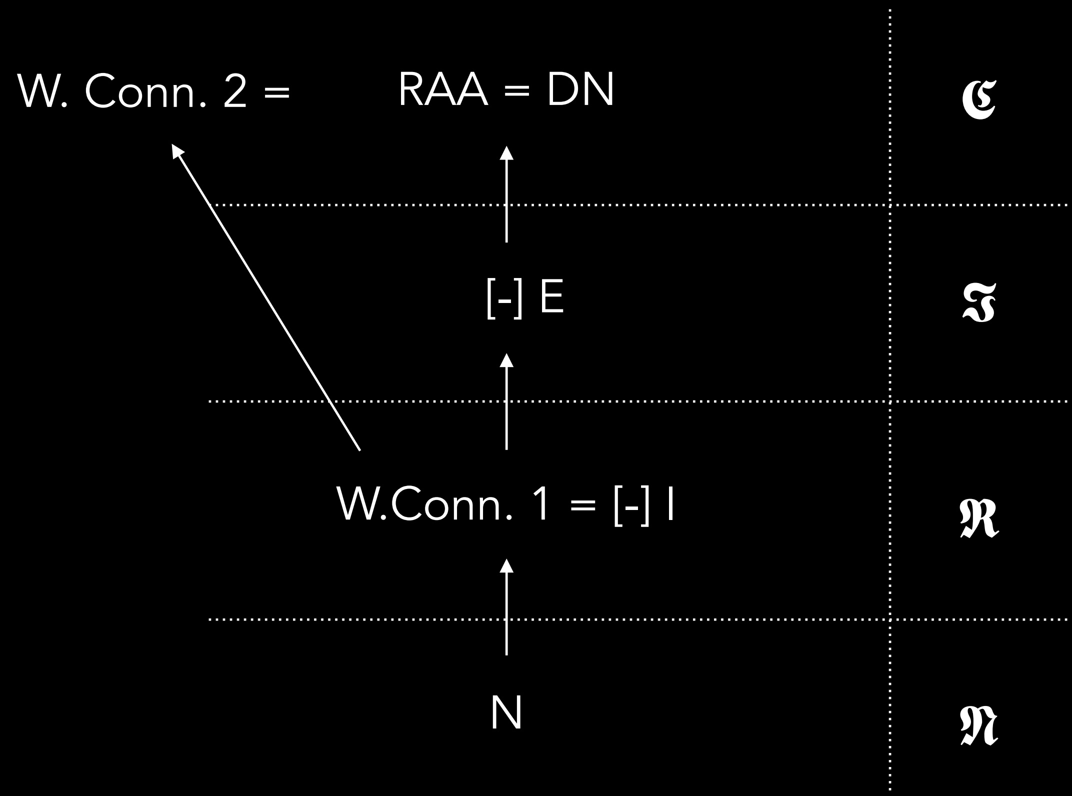
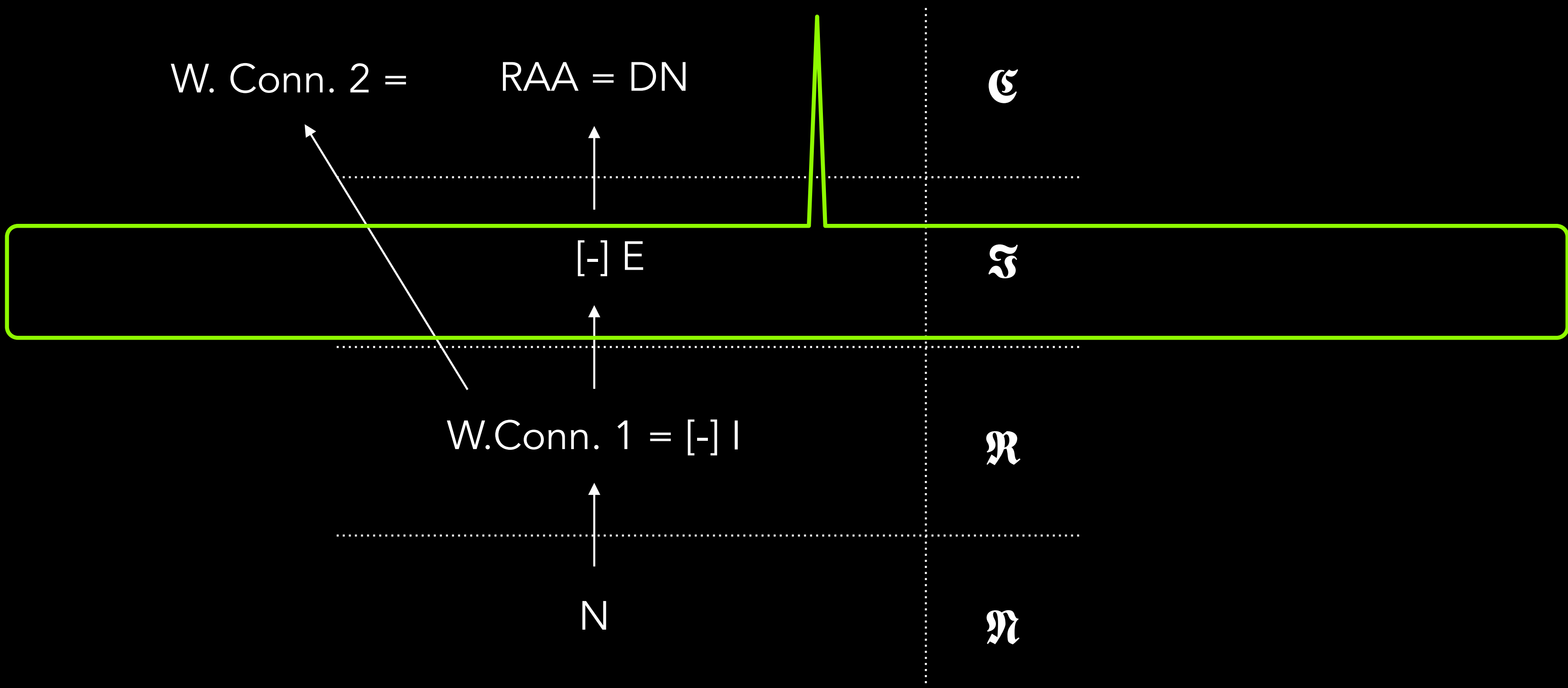
The logic of accept & reject



Sound and complete probabilistic semantics?

The logic of accept & reject

With the accept & reject framework we are here, hence essentially we are dealing with an **intuitionistic** framework



Sound and complete probabilistic semantics?

Characterisation of closure classes

Given a set of signed assessment Γ_s, we define $\Gamma_s^{\diamond} := \{g \in \mathscr{L} \mid [\diamond]g \in \Gamma_s\}$, for $\diamond = +, -$.

- *Theorem*: Let Γ_s be a set of signed assessment, and assume
  - it is $\mathfrak{D}$-consistent (i.e. $\mathrm{Cn}_{\mathfrak{D}}(\Gamma_s^+) \in \mathbf{C}_{\mathfrak{D}}$) and
  - $\mathrm{Cn}_{\mathfrak{D}}(\Gamma_s^+) \cap \Gamma_s^- = \emptyset$.

  Then it holds that
  - Γ_s is $\mathfrak{F}$-consistent,
  - the set of its positive consequences is $(\mathrm{Cn}_{\mathfrak{F}}(\Gamma_s))^+ = \mathrm{Cn}_{\mathfrak{D}}(\Gamma_s^+)$ and
  - the set of its rejected consequences is $(\mathrm{Cn}_{\mathfrak{F}}(\Gamma_s))^- = (\mathrm{Cn}_{\mathfrak{R}}(\Gamma_s))^- \cup (\mathrm{Cn}_{\mathfrak{R}}(\Gamma_s))^- - \mathrm{Cn}_{\mathfrak{D}}(\Gamma_s^+))$.

This result can be seen as the logical analogous (with TDG in the background) of the following characterisation result in (Quaeghebeur, De Cooman & Hermans 2015):

> *Proposition* 2.7. Given a deductively closed assessment without confusion $\mathcal{D}$ in $\mathbb{D}$, then its reckoning extension is a model without confusion: $\mathrm{ext}_{\mathbb{M}}\mathcal{D} = \langle \mathcal{D}_{\succeq}; \overline{\mathcal{D}_{\prec}} \cup (\overline{\mathcal{D}_{\prec}} - \mathcal{D}_{\succeq}) \rangle \in \mathbb{M}$.

- *Theorem*: Let Γ_s be a set of signed assessment, and assume
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Kripke semantics for accept & reject systems

We consider Kripke-like semantics, where each possible word corresponds to a set of lexicographic probabilities, represented as stochastic matrices (of full rank) - and thus, via lexicographic duality, as coherent cones of gambles.

Kripke semantics for accept & reject systems

Given a poset $(W, \leq)$, by $\mathscr{U}(W)$ we denote the set of all upward closed subset of W with respect to ≤: if ($w \in U$ and $w \leq v$) then $v \in U$

A Kripke frame is a triple $F := (W, \leq, N)$ where

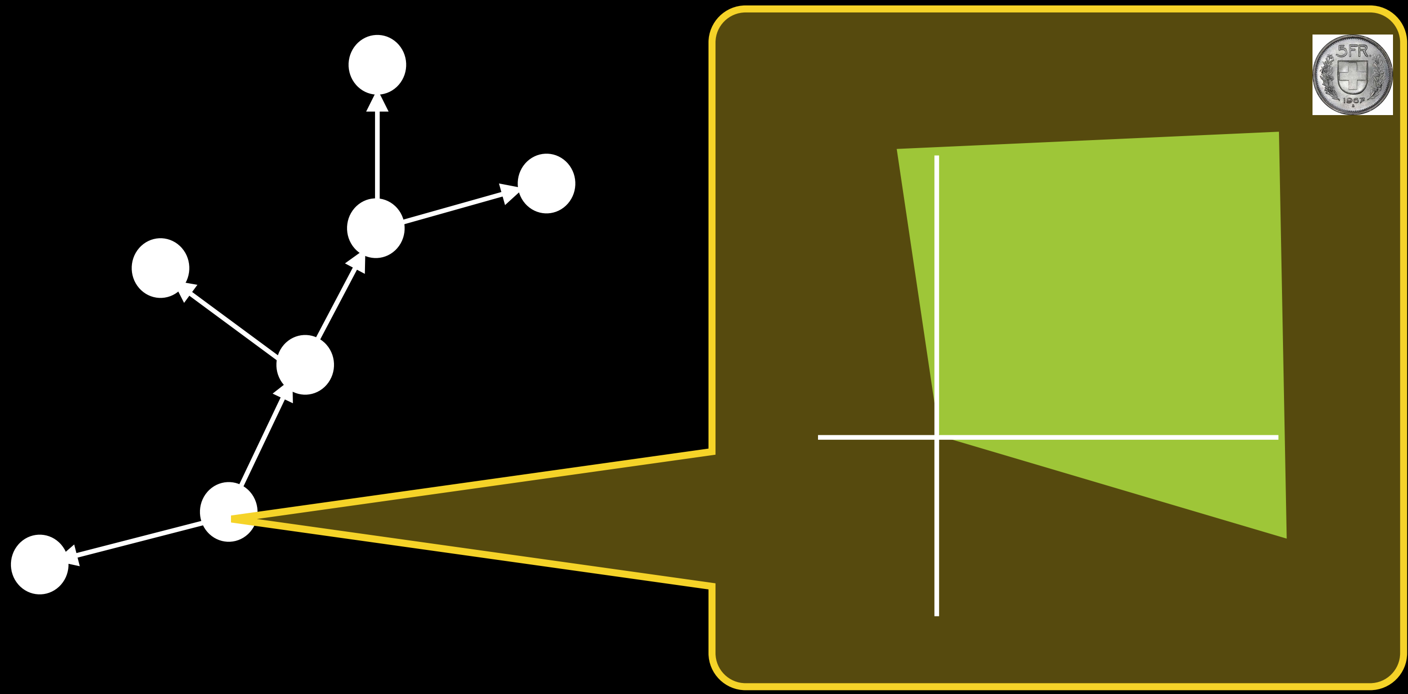
- $(W, \leq)$ is a partial order,
- $N : \mathscr{U}(W) \rightarrow \mathscr{U}(W)$

With this function we will provide a semantics for rejection

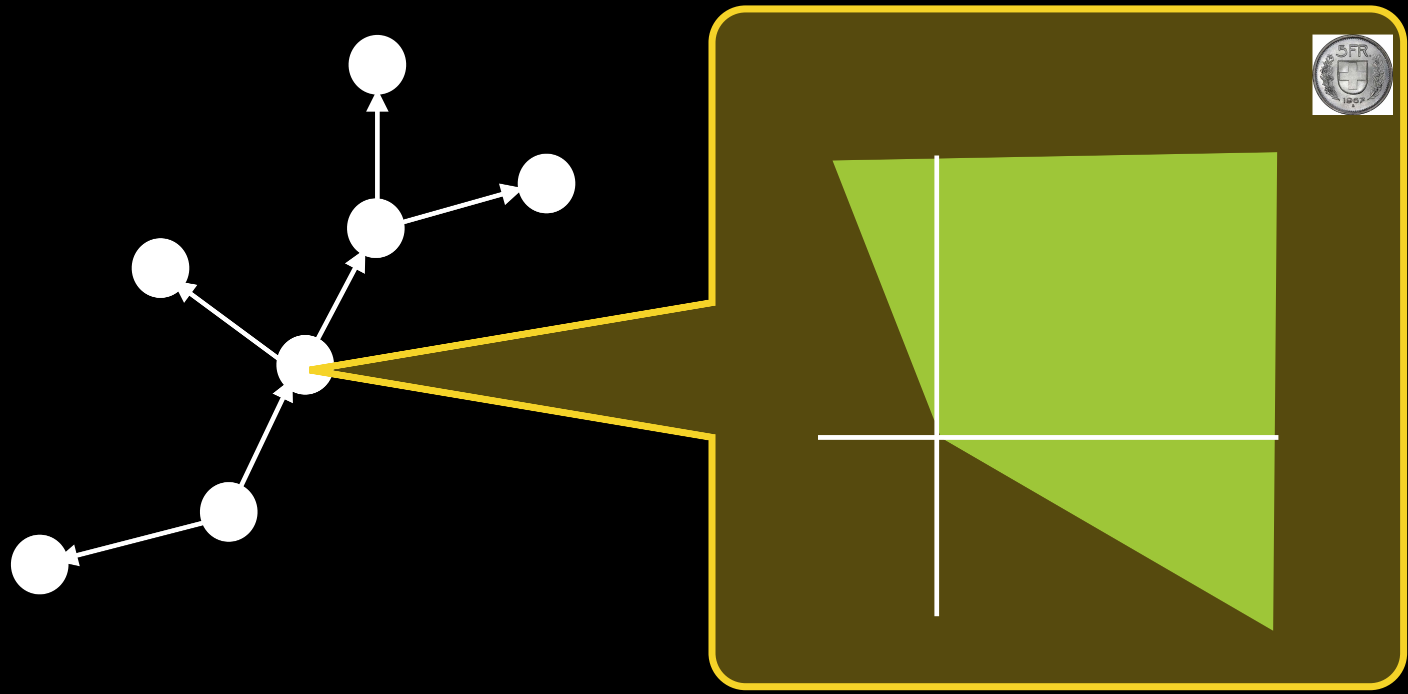
A Kripke model is a pair $\mathfrak{M} := (F, V)$ where

- F is a Kripke frame,
- $V : W \rightarrow \wp(\mathbb{S}(\mathscr{L}))$ is a persistent valuation, i.e. such that if $V(w) \supseteq V(v)$, whenever $w \leq v$

Kripke semantics for accept & reject systems: persistency of valuations:



Kripke semantics for accept & reject systems: persistency of valuations:



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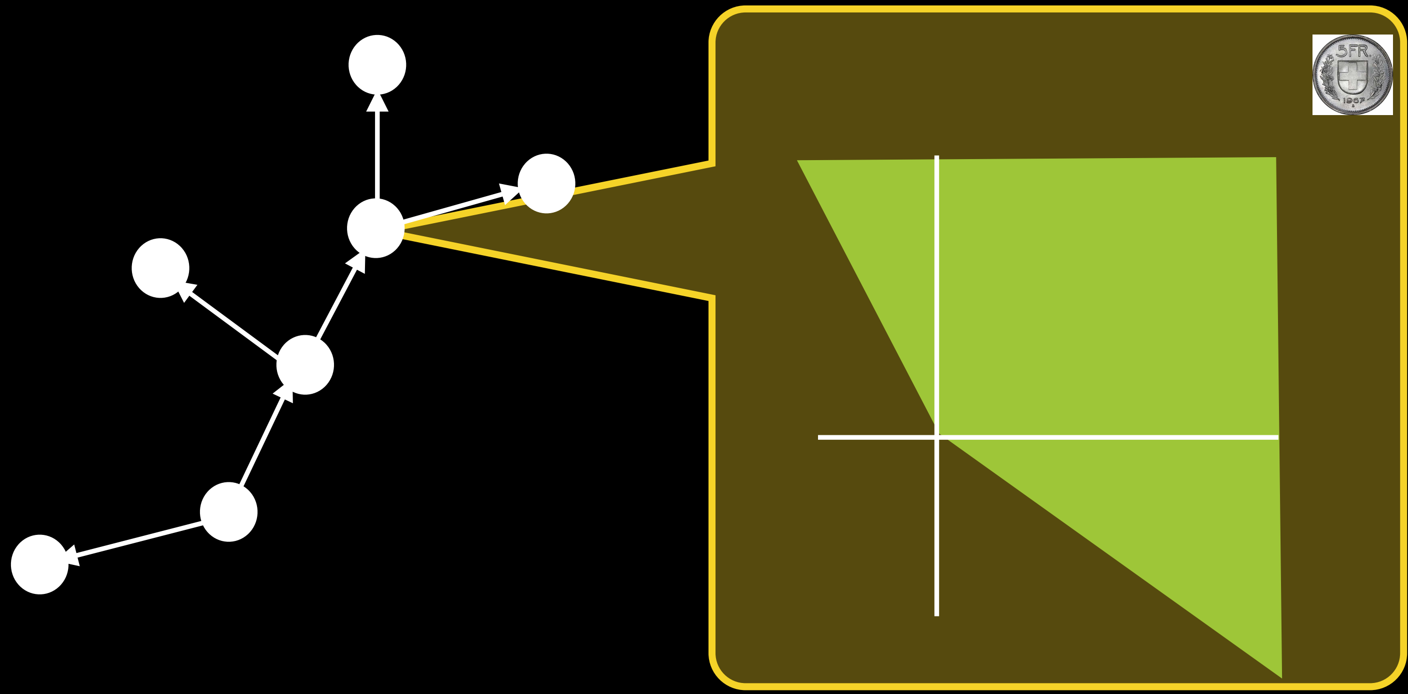

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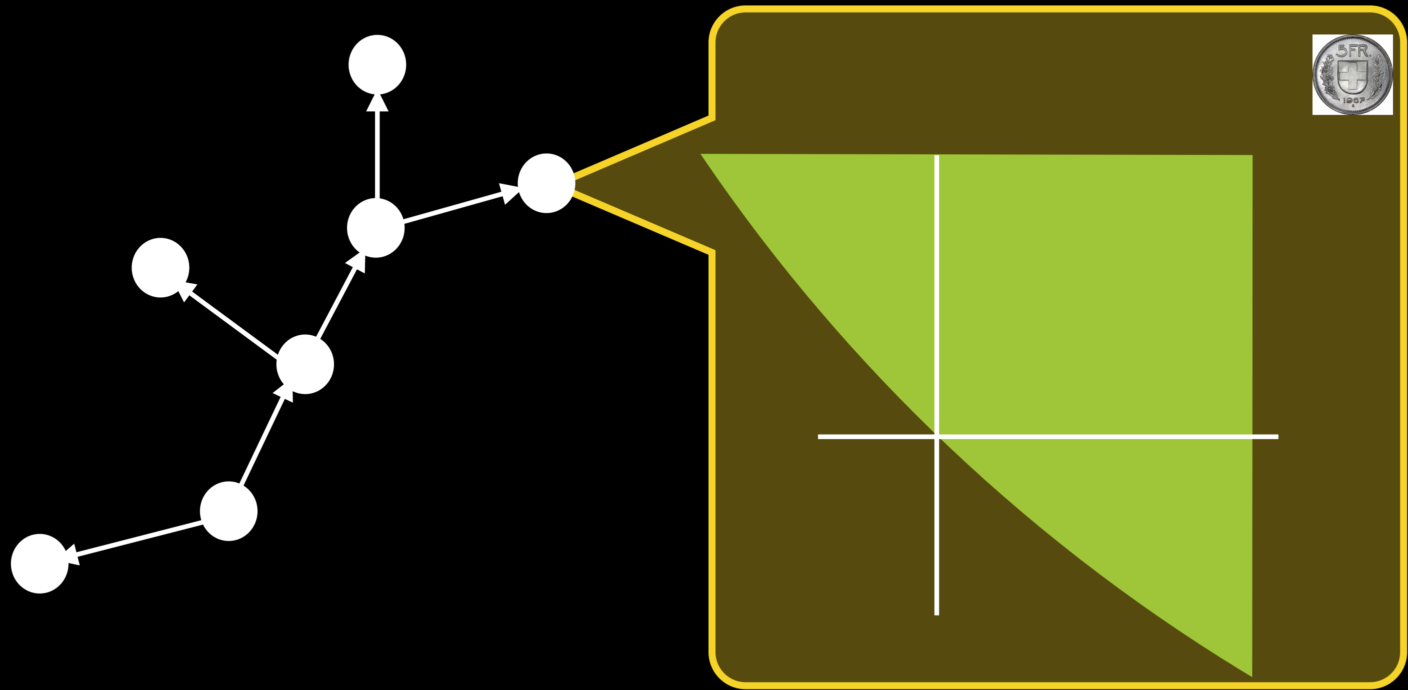


Kripke semantics for accept & reject systems

Let $\mathfrak{M} = (\mathcal{F}, V)$ be a Kripke model and $w \in W$

- $\mathfrak{M}, w \Vdash [+]g$ iff $P \Vdash g, \forall P \in V(w)$

Notice that that $[\![[+]g]\!] := \{w \in W : \mathfrak{M}, w \Vdash [+]g\} \in \mathcal{U}(W)$ by persistency of V. Hence:

- $\mathfrak{M}, w \Vdash [-]g$ iff $w \in N([\![[+]g]\!])$

Given a class $\boldsymbol{X}$ of Kripke model and $\Gamma_s \triangleright [*]g$, we write

$\Gamma_s \vDash_X [*]g$

$\Updownarrow$

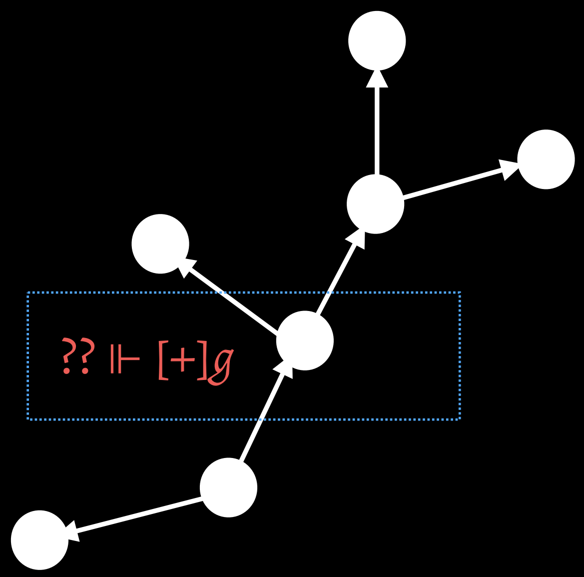
$\forall \mathfrak{M} \in \boldsymbol{X}, \forall w \in W \, (w \in [\![\Gamma_s]\!] \Rightarrow w \in [\![[*]g]\!])$

Kripke semantics for accept & reject systems

- Let $\mathfrak{M} := (F, V)$ be a Kripke model, and consider $w \in W$.
- We set that
  - $\mathfrak{M}, w \models [+]g$ if and only if $P \models g, \forall P \in V(w)$ if and only if $P(g) >_L 0, \forall P \in V(w)$

$g \in (V(w))^{\blacktriangledown}$

Kripke semantics for accept & reject systems: truth in a possible world



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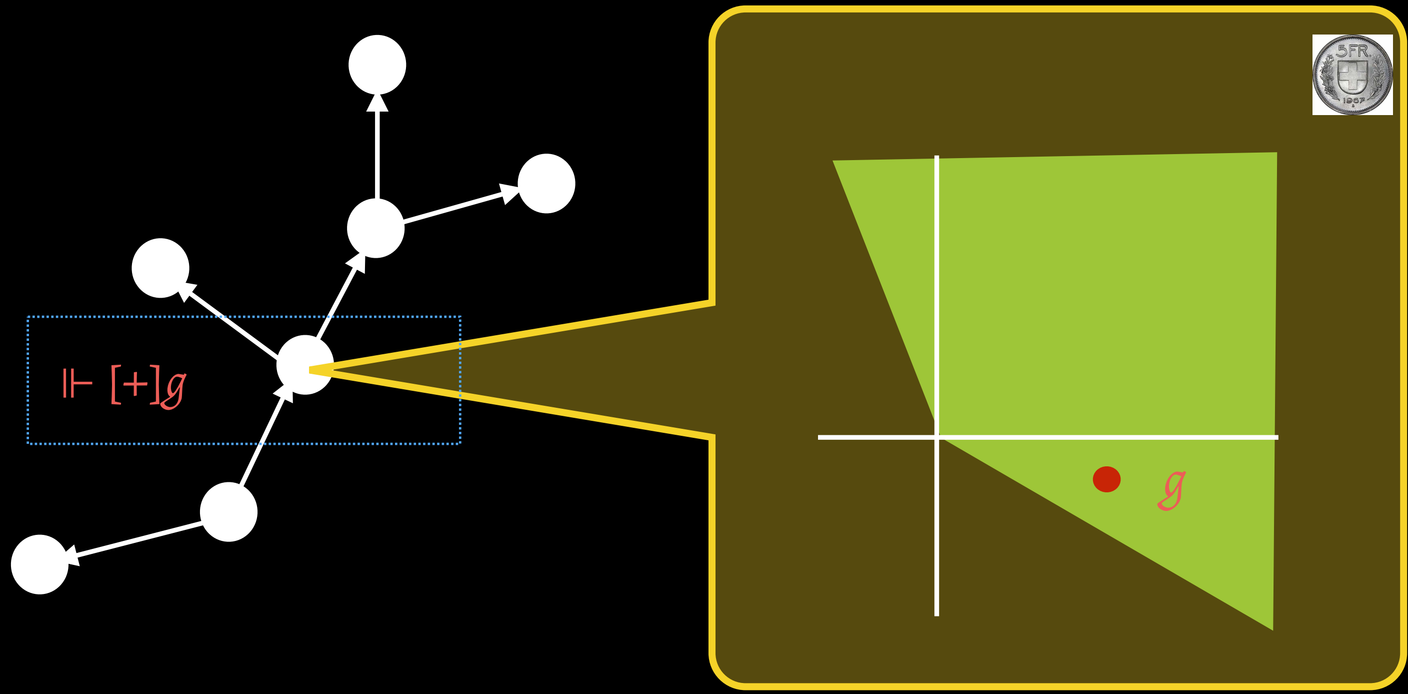

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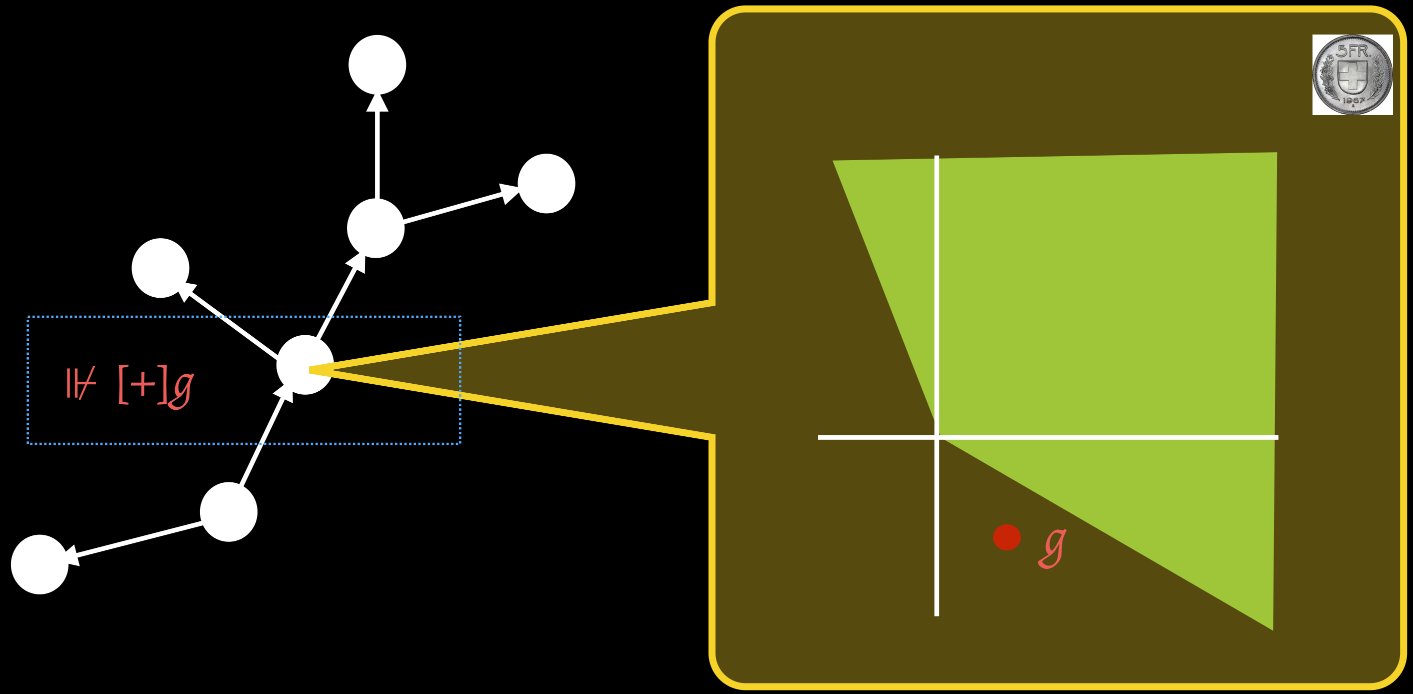

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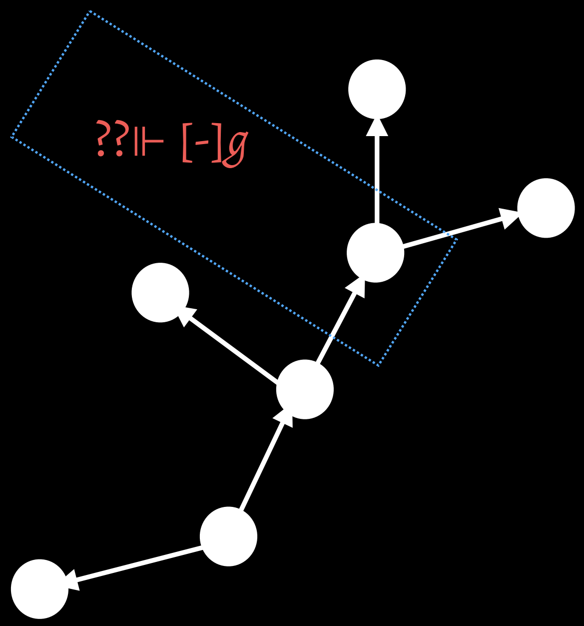
- Notice that that, by persistency of V:

$$\|[+]g\| := \{w \in W \mid \mathfrak{M}, w \vDash [+]g\} \in \mathcal{U}(W)$$

- Hence we set
  - $\mathfrak{M}, w \vDash [-]g$ iff $w \in N(\|[+]g\|)$

Kripke semantics for accept & reject systems: truth in a possible world



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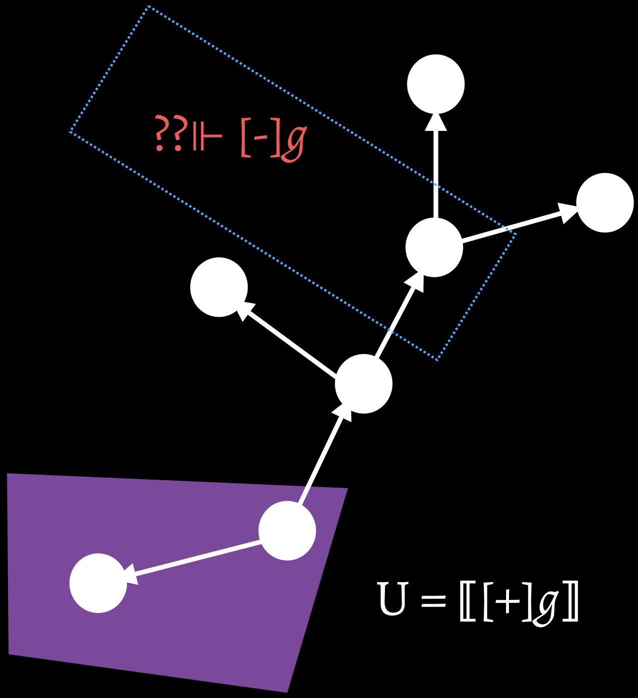

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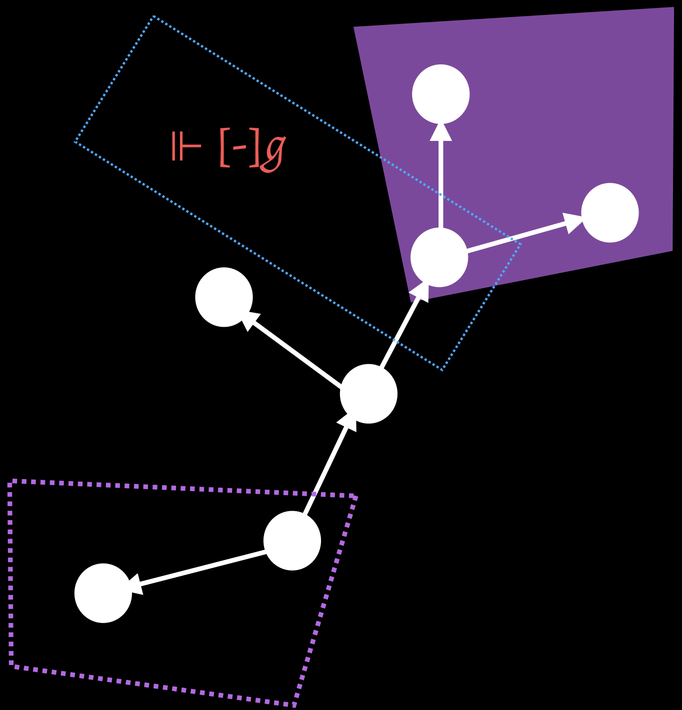
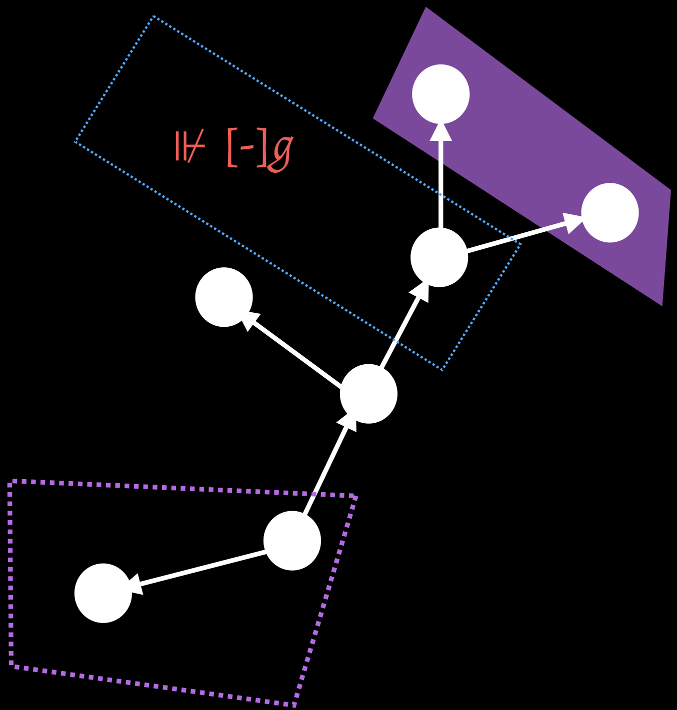


N(U) = N(⟦[+]g⟧)

Kripke semantics for accept & reject systems: truth in a possible world



$$N(U) = N(\llbracket [+]g \rrbracket)$$

Kripke semantics for accept & reject systems: semantic consequence

- Definition: Given a class $\mathfrak{X}$ of Kripke model and a signed sequent $\Gamma_s \rhd [\circ]g$ we write

$$\Gamma_s \vdash_{\mathfrak{X}} \rhd g$$

if and only if for every $\mathfrak{M} \in \mathfrak{X}$ and every $w \in W$, $w \in \|\Gamma_s\|_{\mathfrak{M}}$ implies $w \in \|[\circ g]\|_{\mathfrak{M}}$

Conditions on function N

(P1) $N(U) \cap V = N(U \cap V) \cap V$

(P2) if $U \cap V = \emptyset$, then $U \subseteq N(V)$

(P3) $N(U)$ is the greatest element in $\mathcal{U}(W)$ which is included in the complement of U

Notice that that

- (P3) implies (P1) and (P2)
- (P1) and (P2) are logically independent

Conditions on function N and corresponding systems

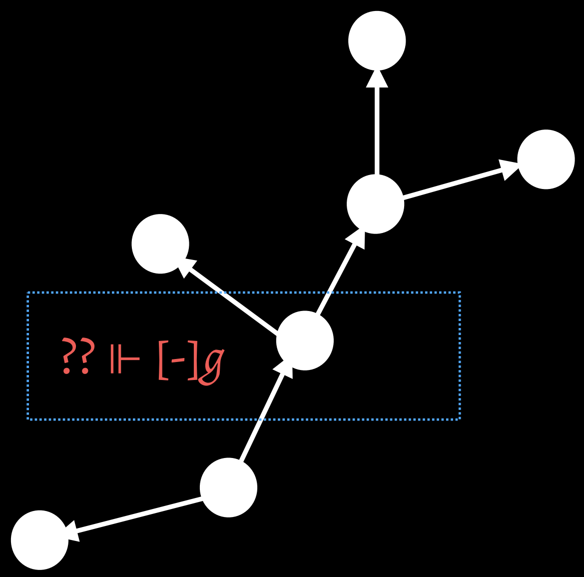
Class of models	*Calculus*	*Rules*
P1	$\mathfrak{N}$	N
P1 + P2	$\mathfrak{R}$	N + [-]I
P3	$\mathfrak{I}$	(N +) [-]I + [-]E
singleton	$\mathfrak{C}$	(N +) [-]I + [-]E + DN

Conditions on function N and corresponding systems

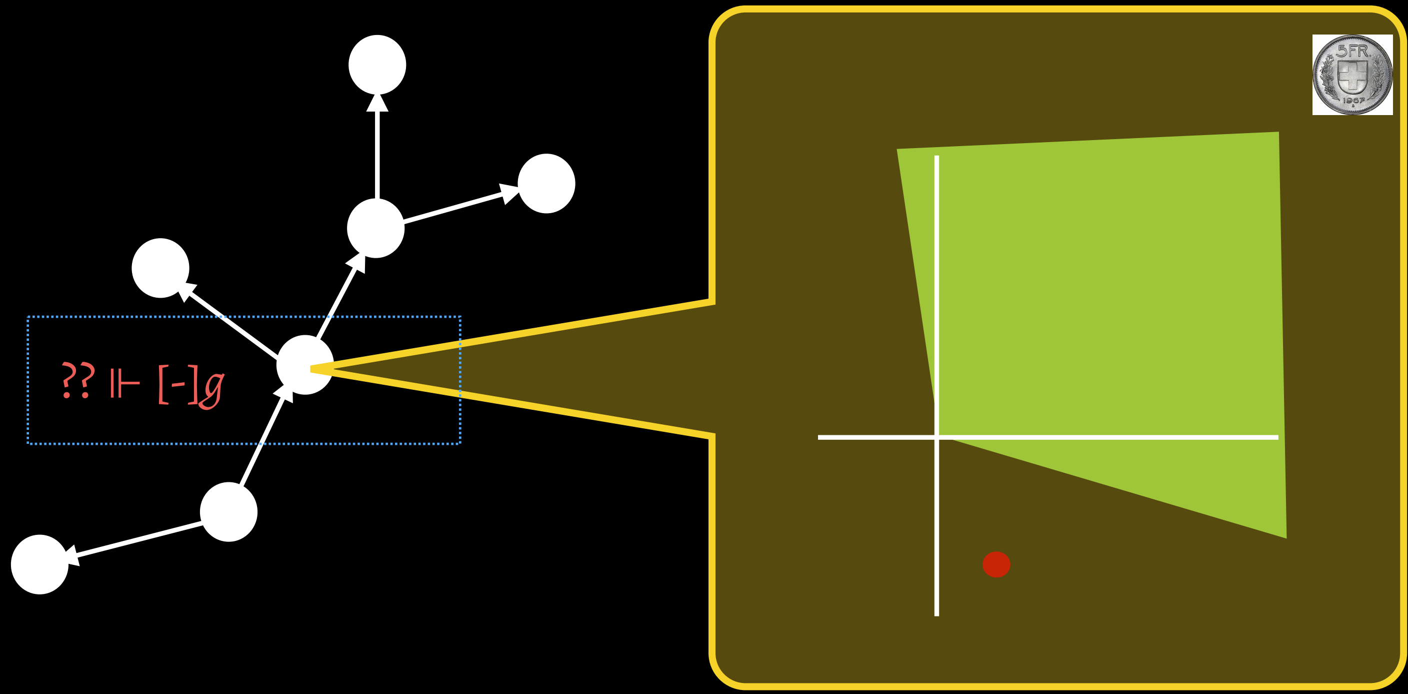
(*"intuitionistic" rejection*) $\mathfrak{M},w \Vdash [-]g$ iff $\forall v \geq w : v \notin [\![[+]g]\!]$

Class of models	*Calculus*	*Rules*
P1	$\mathfrak{N}$	N
P1 + P2	$\mathfrak{R}$	N + [-]I
P3	$\mathfrak{I}$	(N +) [-]I + [-]E
singleton	$\mathfrak{C}$	(N +) [-]I + [-]E + DN

Intuitionistic rejection



Intuitionistic rejection



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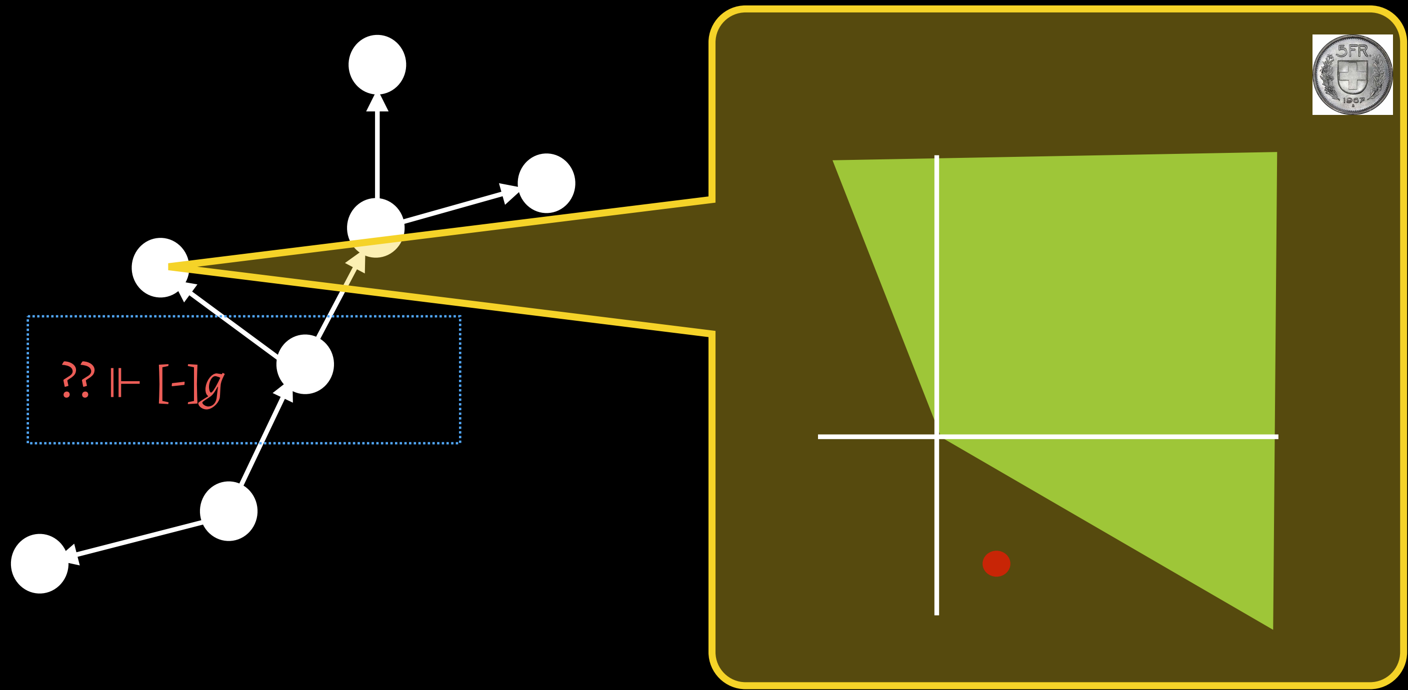

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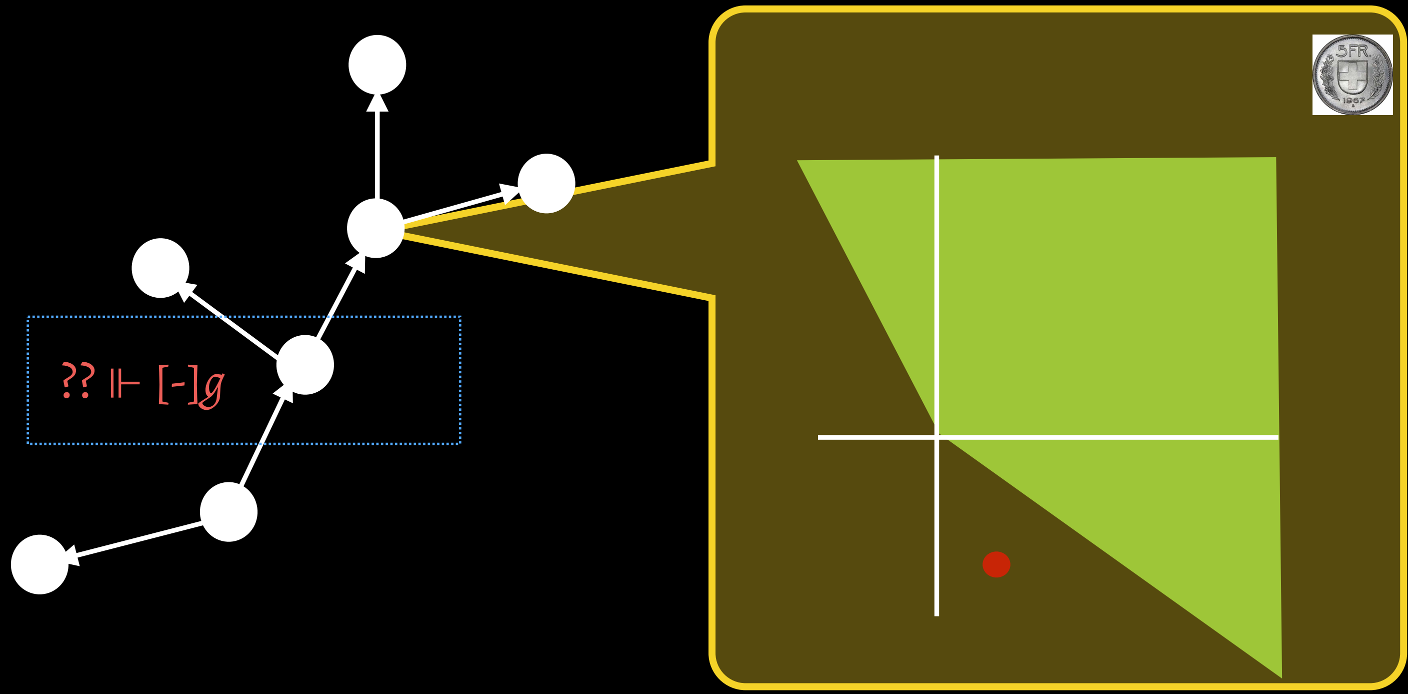

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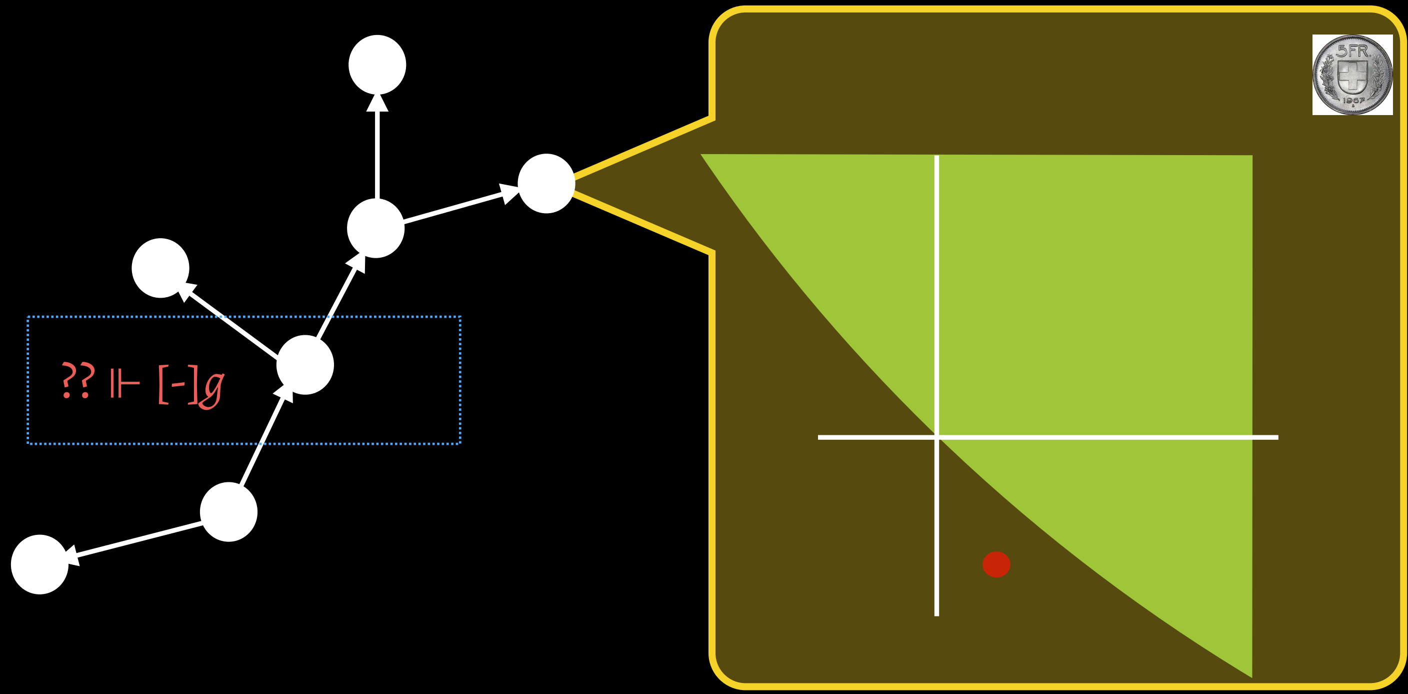

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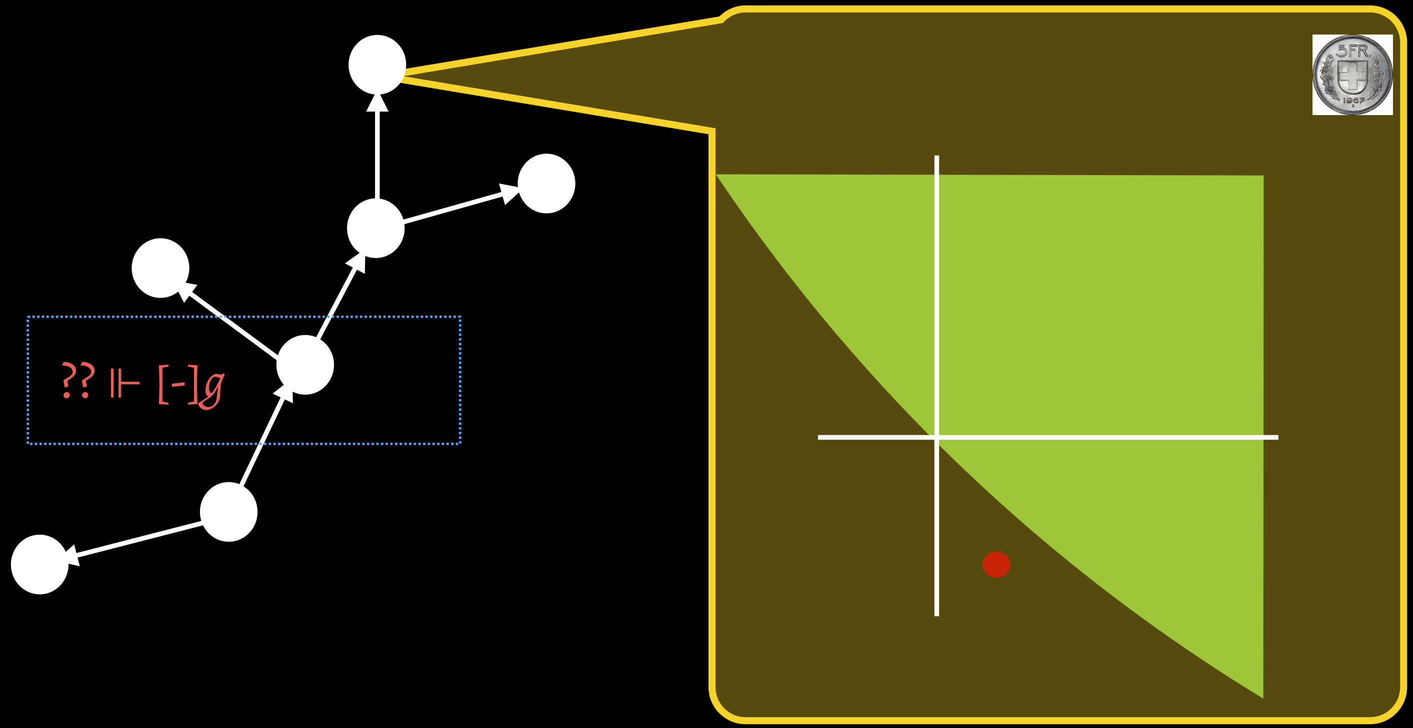
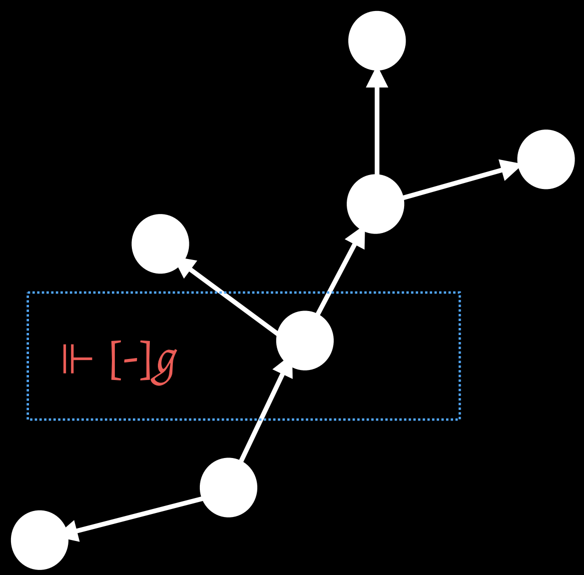


Intuitionistic rejection



Thus, a reading:

- for Alice to reject *g* it means that she is not going to accept *g* in any epistemically plausible situation (w.r.t. the actual one).

(*classical semantics*)

$P \vDash [+]g$ iff $P \Vdash g$ iff $g \in P^{\blacktriangledown}$

$P \vDash [-]g$ iff $P \nvDash [+]g$ iff $P \nVdash g$ iff $g \notin P^{\blacktriangledown}$

$\Gamma_s \vDash [*]g$ iff $\forall P \in \mathbb{S}$: $(P \vDash \Gamma_s \Rightarrow P \vDash [*]g)$

Classical rejection

Class of models	*Calculus*	*Rules*
P1	$\mathfrak{N}$	N
P1 + P2	$\mathfrak{R}$	N + [-]I
P3	$\mathfrak{I}$	(N +) [-]I + [-]E
singleton	$\mathfrak{C}$	(N +) [-]I + [-]E + DN

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Conclusions

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Conclusion

- The theory of desirable gambles (TDG) **is** a logic
- We can study the act of accepting and rejecting gambles from a logical point of view
  - It is thus natural to go further…

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What next

- extend the proposed logical approach to Gert & Erik framework for "reject & accept" by adding specific Gentzen-type rules for conjunction and disjunction;
- study the properties of the system(s)
- verify if and how it is possible to link them with (the desirability view on) choice functions (or not?)

- What about
  - Is there ar … at can be added to the fran …
- What about adding other connectives?
  - Can we capture within these extension the theory of choice functions?
  - And what about the theory of things? Should we also change the type of consequence relation?

Conjunction and disjunction

Conjunction and disjunction

- Let $(A, \vdash)$ be some consequence system, we say that
  - a binary function $\wedge : A \times A \to A$ is a **conjunction** if for every $a, b \in A$ and every $\Gamma \subseteq A$
    - $\Gamma, a \wedge b \vdash a$ and $\Gamma, a \wedge b \vdash b$ (∧-elimination)
    - $\Gamma, a, b \vdash a \wedge b$ (∧-introduction)
  - a binary function $\vee : A \times A \to A$ is a **disjunction** if for every $a, b, c \in A$, and every $\Gamma \subseteq A$
    - if $\Gamma, a \vdash c$ and $\Gamma, b \vdash c$, then $\Gamma, a \vee b \vdash c$ (∨-elimination)
    - $\Gamma, a \vdash a \vee b$ and $\Gamma, b \vdash a \vee b$ (∨-introduction)

Conjunction and disjunction

These conditions can straightforwardly been turned into Gentzen-style rules

- Let $(A, \vdash)$ be some consequence system, we say that
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Conjunction and disj

What about when a consequence system has both conjunction and disjunction? What kind of properties should we expect? Remember that in case we are consider a unary implication, and in particular lattices, distributivity does not necessarily hold (see e.g "quantum logic")

- Let $(A, \vdash)$ be some consequence system, we say that
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    - $\Gamma, a \vdash a \vee b$ and $\Gamma, b \vdash a \vee b$ (∨-introduction)

Conjunction and disjunction

The fact of dealing with a “set-thing”-relation has some “unintuitive” consequences, such as:

- *Theorem*: Let $(A, \vdash)$ be some consequence system that has conjunction and disjunction. Then it satisfies the **distribution laws**, that is for every $a, b, c \in A$, and every $\Gamma \subseteq A$
  - $\Gamma, a \wedge (b \vee c) \dashv\vdash \Gamma, (a \wedge b) \vee (a \wedge c)$
  - $\Gamma, a \vee (b \wedge c) \dashv\vdash \Gamma, (a \vee b) \wedge (a \vee c)$

- *Proof*: We just verify the first case. We have that $a \wedge (b \vee c) \vdash a$ and $a \wedge (b \vee c) \vdash b \vee c$. This implies that $a \wedge (b \vee c), b \vdash (a \wedge b)$ and thus $a \wedge (b \vee c), b \vdash (a \wedge b) \vee (a \wedge c)$. Similarly $a \wedge (b \vee c), c \vdash (a \wedge b) \vee (a \wedge c)$. By cut and ∨-elimination, we get that $a \wedge (b \vee c) \vdash (a \wedge b) \vee (a \wedge c)$. For the other direction, it is immediate to first check that by ∨-introduction $(a \wedge b) \vdash a \wedge (b \vee c)$ and $(a \wedge c) \vdash a \wedge (b \vee c)$, and then, as before, conclude by ∨-elimination.

Conjunction and disjunction

The fact of dealing with a "set-thing"-relation has some "unintuitive" consequences, such as:

- *Theorem*: Let $(A, \vdash\ )$ be some consequence system that has conjunction and disjunction. Then it satisfies the

  -

  -

- *Proo* ... that $a \wedge$ ... $(a \wedge c)$.
  By c ... nediate to fi ... ore, conc

For more on this "paradoxical" issue, see:

A Paradox in the Combination of Logics

Jean-Yves Béziau*

In this paper we present a fact, surprising enough to be called a paradox, which shows that the central issue in combination of logic is still problematic. This issue has been described by Dov Gabbay in his book on fibration as follows "Combine $S1$ and $S2$ into a system S which is the smallest logical system for the combined language which is a conservative extension of both $S1$ and $S2$. The two systems are presented in totally different ways. How are we going to combine them." ([2], p.7)

Given two logics $L1$ and $L2$, let us call $L1 * L2$ the combination of $L1$ and $L2$ described by Gabbay, i.e. the smallest logic for the combined language which is a conservative extension of both $L1$ and $L2$. If we have a mechanism for combining semantics or proof systems, how can we be sure that this mechanism produces $L1 * L2$? If we have a technique to combine a Kripke semantics $K1$ generating a logic $L1$ and a Kripke semantics $K2$ generating a logic $L2$, we would

Combining Conjunction with Disjunction

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Abstract. In this paper we address some central problems of combination of logics through the study of a very simple but highly informative case, the combination of the logics of disjunction and conjunction. At first it seems that it would be very easy to combine such logics, but the following problem arises: if we combine these logics in a straightforward way, distributivity holds. On the other hand, distributivity does not arise if we use the usual notion of extension between consequence relations. A detailed discussion about this phenomenon, as well as some possible solutions for it, are given.

Three main logics

Classical logic

conjunction, disjunction and classical negation

↑

Intuitionistic logic

conjunction, disjunction and intuitionistic negation

↑

Minimal logic

conjunction, disjunction and minimal negation

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Some additional basics of lattice theory

Filters …

- *Definition*: Let $(A, \wedge, \vee)$ be a lattice. A subset $F \subseteq A$ is a **filter**, if, for every $a, b \in A$, the following hold

  (F1) $a \in F$ and $b \in F$ implies $a \wedge b \in F$

  (F2) $a \in F$ or $b \in F$ implies $a \vee b \in F$

  The second condition is sometimes expressed in the following equivalent form

  (F2′) $a \in F$ and $a \leq b$ implies $b \in F$

  A filter $F \subseteq A$ is said to be

  - **proper** whenever $F \neq A$,
  - **prime**, whenever it is proper and: $a \vee b \in F$ implies either $a \in F$ or $b \in F$, for every $a, b \in A$,
  - **maximal** (or a **ultrafilter**) if it is proper and if $F' \supsetneq F$ is a filter, then $F' = A$.

Filters …

- *Fact*: Let $(A, \wedge, \vee)$ be a lattice. Consider a non empty collection $\mathscr{F} \subseteq \wp(A)$ of filters. Then $\bigcap \mathscr{F} \subseteq A$ is also a filter. In particular, this means that, given a set $B \subseteq A$, there is the smallest filter $[B) := \bigcap \{F \subseteq A \mid F \supseteq B \text{ and } F \text{ is a filter.}\}$ extending B. Moreover, it holds that

$$[B) = \{a \in A \mid \exists b_1, \ldots, b_n \in B \text{ s.t. } b_1 \wedge \ldots \wedge b_n \leq a\}$$

Whenever a filter $F = [B)$, we say that B generates F.

- *Definition*: Let $(A, \wedge, \vee)$ be a lattice. Whenever a filter $F = [B)$, we say that B generates F. When B is a singleton, we say that F is a **principal filter**.

Ultrafilters

- *Theorem (ultrafilter extension principle)*: Let $(A, \wedge, \vee)$ be a lattice. Any proper filter $F \subseteq A$ is contained in a ultrafilter $F^{+} \supseteq F$.

  Proof: Consider the family $\mathscr{F} \subseteq \wp(A)$ of proper filters extending F ordered by inclusion. Clearly $\mathscr{F}$ contains F. Consider now any chain $\mathscr{C}$ of proper filters extending F. Notice that $F \cup \bigcup \mathscr{C}$ is a proper filter extending F and an upper bound of $\mathscr{C}$. By Zorn's lemma there is a proper filter maximal among those that extend F, which is therefore maximal among all proper filters

Being separated by (prime) filters

- *Theorem (prime filter separation principle)*: Let $(A, \wedge, \vee)$ be a lattice, and let $a, b \in A$ such that $a \nleq b$. Then there is a ultrafilter $F \subseteq A$ such that $a \in F$ but $b \notin F$.

  Proof: Consider the principal filter $[a)$. Clearly $b \notin [a)$, but it does not need to be prime (i.e. a ultrafilter). As before, take any chain $\mathscr{C}$ of proper filters extending $[a)$ an such that $b \notin F$ for every $F \in \mathscr{C}$. Notice that $\bigcup \mathscr{C}$ is a proper filter extending $[a)$ such that $b \notin \bigcup \mathscr{C}$ and is an upper bound of $\mathscr{C}$. By Zorn's lemma there is a proper filter $F' \subseteq A$ maximal among those that extend $[a)$ and do not contain b. One just then check that F' is actually prime.

Distributive lattices

- *Definition*: A lattice $(A, \wedge, \vee)$ is said to be distributive whenever it satisfies the following identities:
  - $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
  - $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

- *Fact*: In a distributive lattice, any ultrafilter is necessarily prime, although not every prime filter is maximal.

Stone's representation theorem for distributive lattice

- *Theorem (Stone's representation theorem)*: Every distributive lattice is isomorphic to a ring of sets (i.e. a collection of sets closed under binary intersection and union).

  Proof: Obviously every ring of sets is a distributive lattice. So let $(A, \wedge, \vee)$ be a distributive lattice. Define the ring of sets (lattice) $\mathscr{R} := (\{\{P \subsetneq A \mid a \in P \text{ is a prime filter}\} \mid a \in A\}, \cap, \cup)$. The map $h : a \mapsto \{P \subsetneq A \mid a \in P$ is a prime filter$\}$ is one-to-one by the prime filter separation principle. One then check that it actually defines a homomorphism from $(A, \wedge, \vee)$ onto $\mathscr{R}$.

Ideals

- *Definition*: Let $(A, \wedge, \vee)$ be a lattice. A subset $I \subseteq A$ is a **ideal**, if, for every $a, b \in A$, the following hold

  (I1) $a \in I$ and $b \in I$ implies $a \vee b \in I$

  (I2) $a \in I$ or $b \in I$ implies $a \wedge b \in I$

  The second condition is sometimes expressed in the following equivalent form

  (I2') $a \in I$ and $a \geq b$ implies $b \in I$

  A ideal $I \subseteq A$ is said to be

  - **proper** whenever $I \neq \emptyset$,
  - **prime**, whenever it is proper and: $a \wedge b \in I$ implies either $a \in I$ or $b \in I$, for every $a, b \in A$,
  - **maximal** (or a **ultrafilter**) if it is proper and if $F' \supsetneq F$ is a filter, then $F' = A$.
  - **principal** if $I = \{a \in A \mid a \leq i\}$, for some $i \in A$, and we denote it by $(i]$.

Filters and ideals

- Seeing a partial order as "logical entailment", a filter can be thought of as a theory, that is a logically closed set of propositions/claims / assertions;
  - as such it behaves exactly like a classical truth set (possible world) with respect to conjunction, and halfway like a classical truth set with respect to disjunction.
- What make a filter behaves exactly like a classical truth set, is the converse of F2, that is the defining condition of a prime filter
  - thus in a prime filter (prime theory, prime world), the conjunction of two propositions is true if and only if both of the propositions are true, and the disjunction of two propositions is true fi and only if at least one of the propositions is true.
- Similarly, an ideal can be thought of as a counter-theory, i.e., a logically closed collection of disclaimers, and a prime ideal can be thought of as a"false ideal".