Introduction to imprecise probabilities

SIPTA School 2024, Ghent, Belgium

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Overview

Kick-off (slot 1)

Classical probability theory (slot 1)

Interpretation of probability (slot 2)

Limitations of probability theory (slot 2)

Probability intervals (slot 3)

Credal sets (slot 3–4)

Acceptability & Desirability (slot 4–5)

Interval expectation & probability (slot 5–6)

Overview

Kick-off (slot 1)

- Uncertainty, representation & reasoning
- Illustration: Blood groups

Classical probability theory (slot 1)

Interpretation of probability (slot 2)

Limitations of probability theory (slot 2)

Probability intervals (slot 3)

Credal sets (slot 3–4)

Acceptability & Desirability (slot 4–5)

Interval expectation & probability (slot 5–6)

What is *uncertainty*?

There is no consensus definition of uncertainty...

> Uncertainty is any departure
> from the unachievable ideal of complete determinism.
>
> Walker et al. (2003)

> Uncertainty is the lack of certainty, a state of limited knowledge where it is impossible to exactly describe the existing state, a future outcome, or more than one possible outcome.
>
> Hubbard (2014)

How can we *represent* uncertainty?

Uncertainty representations are defined by axioms

> Any reasonable measure of belief is isomorphic to a probability distribution.
>
> Cox (1946), paraphrased

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Some concrete examples of representations

- probability distributions
- intervals, sets
- possibility distributions, belief functions
- sets of desirable gambles, preference orders
- lower/upper probabilities/expectations
- credal sets
- choice functions

What is *reasoning* under uncertainty?

Reasoning under uncertainty: deducing inferences or making decisions

> In each problem situation, uncertainty is associated with some purpose.
> It may, for example, be associated with prediction, retrodiction, prescription, or decision making.
>
> Klir (1995)

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Some concrete examples

- Predicting individual disease risk based on medical history
- Planning and designing of blood bank for a hospital with 100 beds
- A newspaper vendor must decide how many copies to purchase each day

Blood groups problem with expensive test

Problem setup

- A sample:

A	AB	O	B
A	B	A	O

- A disease

- Two treatments with differing effectiveness:

	A	B	AB	O
f_1	0.5	0.6	0.7	0.1
f_2	0.4	0.3	0.3	0.8

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Representation

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Representation

PMF from observed frequencies:

p_A	p_B	p_{AB}	p_O
$3/8$	$2/8$	$1/8$	$2/8$

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Reasoning

- Probability $P(\{\text{A}, \text{O}\})$?
- Expectation $E_p(f_1)$?
- Outcome with maximal probability?
- Treatment with highest expected effectiveness?

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Reasoning

- $P(\{\mathrm{A}, \mathrm{O}\}) = 5/8$
- $E_p(f_1) = 0.45$ (*check!*)
- Outcome A has maximal probability
- Treatment 2 has highest expected effectiveness (*check!*)

Blood groups problem with cheap test

Problem setup

- A partial information sample:

A,B	AB	O	A,B
A,B	A,B	A,B	O

- A disease

- Two treatments with differing effectiveness:

	A	B	AB	O
f_1	0.5	0.6	0.7	0.1
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Blood groups problem with cheap test

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?	?	$1/8$	$2/8$

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Representation

PMF from observed frequencies:

p_A	p_B	p_{AB}	p_O
?	?	1/8	2/8

Reasoning

- $P(\{A, O\}) = ?$
- $E_p(f_1) = ?$
- Outcome ? has maximal probability
- Treatment ? has highest expected effectiveness

We want to be able
to represent uncertainty and reason
also in situations
with partial information

Overview

Basic setup and axioms

Basic setup of **probability theory**:

- Random variable X
- Set of possible outcomes $\mathcal{X}$
- Each possible event/set $S \subseteq \mathcal{X}$
  is assigned a **probability value** $P(S)$

Basic setup and axioms

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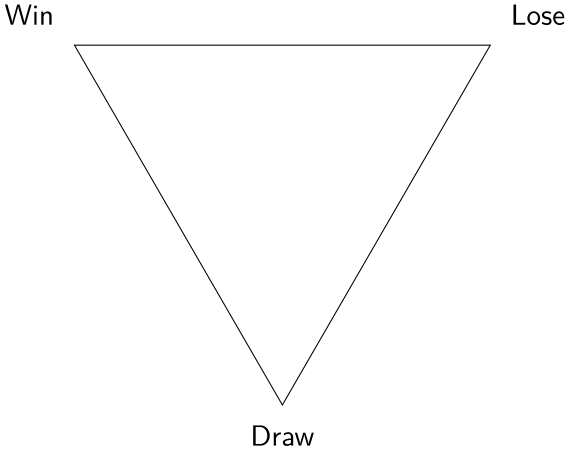
Axioms

A **probability mass function** p (and the corresponding probability measure P) must be:

1. Nonnegative: $p_x \geq 0$ for all outcomes $x \in \mathcal{X}$
2. Additive: $P(S) = \sum_{x \in S} p_x$ for all events $S \subseteq \mathcal{X}$
3. Normed: $P(\mathcal{X}) = 1$

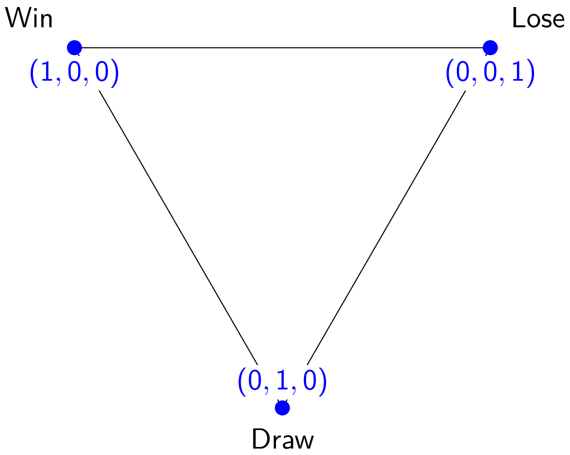
Probability simplex: visualizing probability mass functions

- $\mathcal{X} = \{\text{Win}, \text{Draw}, \text{Lose}\}$



Probability simplex: visualizing probability mass functions

- $\mathcal{X} = \{\text{Win}, \text{Draw}, \text{Lose}\}$
- 'degenerate' probability mass functions (pmfs) $p = (p_\text{W}, p_\text{D}, p_\text{L})$ at the corners



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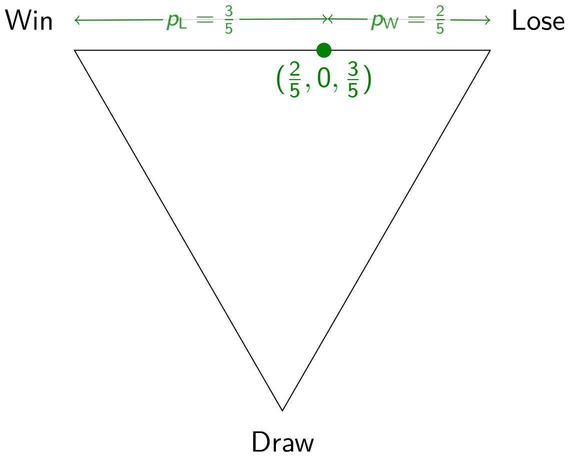
- other pmfs as convex combinations thereof; values can be 'read off' as distance to opposite edge



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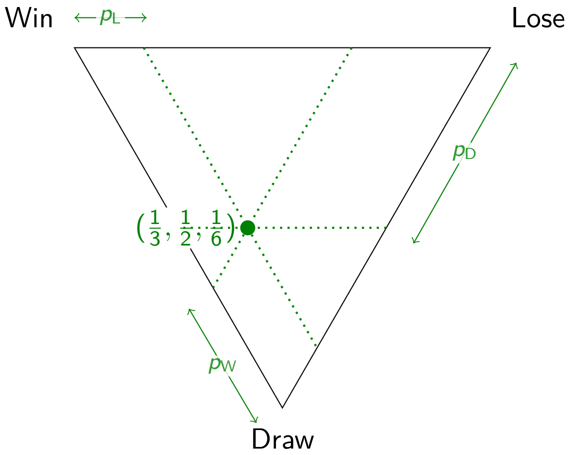
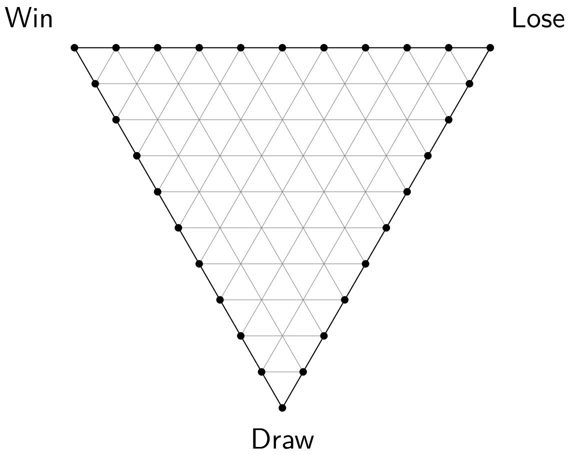


Probability simplex: visualizing probability mass functions — exercise

On a gridded probability simplex on your worksheet, indicate

- The degenerate pmf corresponding to $p_L = 1$
- The pmf $(p_W, p_D, p_L) = (05, 0.5, 0)$
- The pmf $(p_W, p_D, p_L) = (0.1, 0.3, 0.6)$



Reasoning — deducing inferences and making decisions

Deducing inferences

- probability values
- expectations/previsions of real-valued functions f on $\mathcal{X}$:

$$E_p(f) := \sum_{x\in\mathcal{X}} p_x f(x)$$

Reasoning — deducing inferences and making decisions

Deducing inferences

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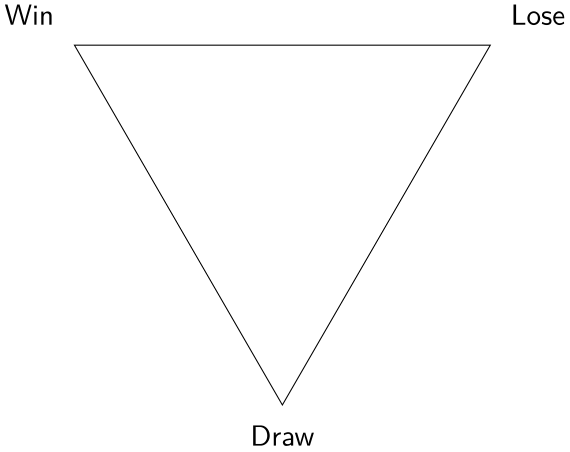
Decision making

- outcomes with maximal probability
- options minimizing/maximizing expectation

- *Topic of tomorrow morning's lecture*

Probability simplex: visualizing probabilities and expectations

- $\mathcal{X} = \{\text{Win}, \text{Draw}, \text{Lose}\}$
- Visualize probability and expectation values as the lines of pmfs for which that value is attained

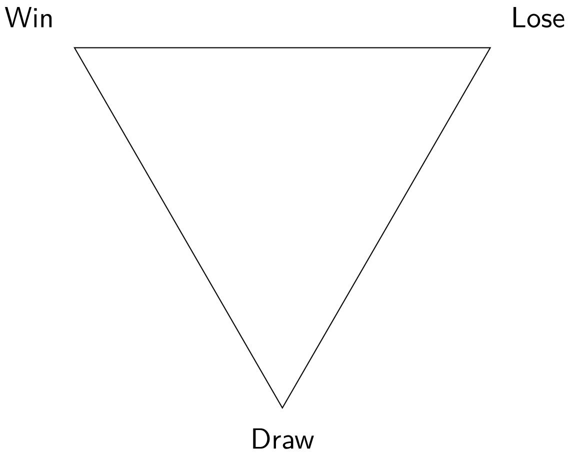


Probability simplex: visualizing probabilities and expectations

- $\mathcal{X} = \{\text{Win}, \text{Draw}, \text{Lose}\}$
- Visualize probability and expectation values as the lines of pmfs for which that value is attained

Probability example

- $A = \{\text{Win}, \text{Lose}\}$
- $P(A) = 2/3$

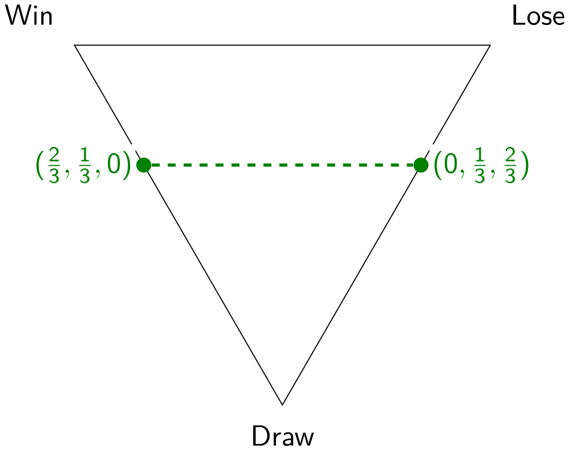


Probability simplex: visualizing probabilities and expectations

- $\mathcal{X} = \{\text{Win}, \text{Draw}, \text{Lose}\}$
- Visualize probability and expectation values as the lines of pmfs for which that value is attained

Probability example

- $A = \{\text{Win}, \text{Lose}\}$
- $P(A) = p_\text{W} + p_\text{L} = 2/3$

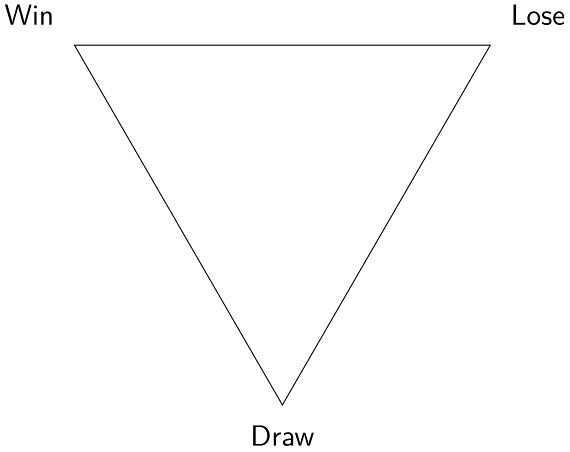


Probability simplex: visualizing probabilities and expectations

- $\mathcal{X} = \{\text{Win}, \text{Draw}, \text{Lose}\}$
- Visualize probability and expectation values as the lines of pmfs for which that value is attained

Expectation example

- $f = (f_{\text{W}}, f_{\text{D}}, f_{\text{L}}) = (1, 0, -1)$
- $E(f) = -1/2$



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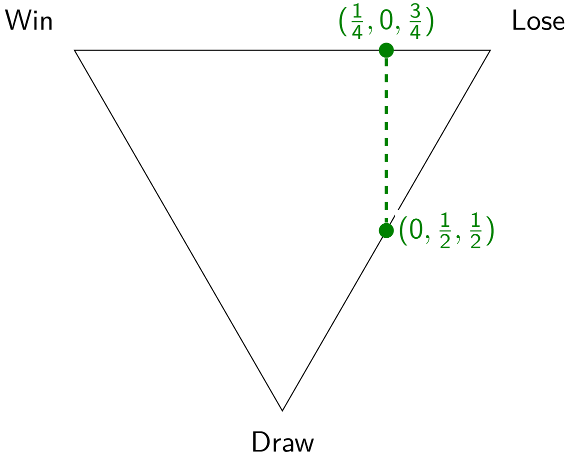


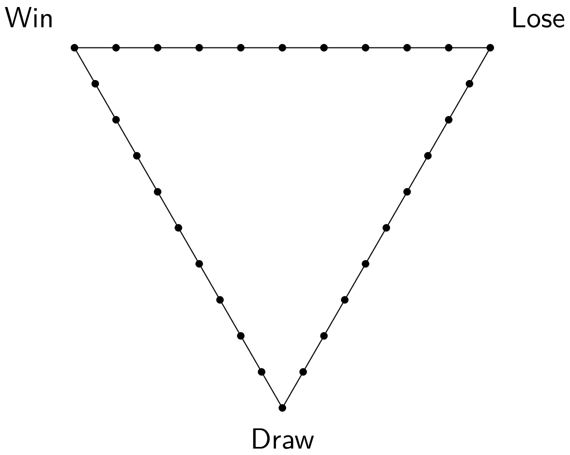
Expectation example

- $f = (f_W, f_D, f_L) = (1, 0, -1)$
- $E(f) = p_W - p_L = -1/2$

Probability simplex: visualizing probabilities and expectations — exercise

On a non-gridded probability simplex on your worksheet, indicate

- The set of pmfs for which $P(\{\mathrm{D}, \mathrm{L}\}) = 0.4$
- The set of pmfs for which $E(f) = 0$ with $f = (-1, -4, 4)$
- Is there a pmf compatible with both?
  (If yes, which?)



Learning — creating a representation

- from data, using estimation techniques (learning)
- from experts, using elicitation (asking questions)

Multivariate probability mass functions: basic setup

- Index set $N = \{1, \dots, n\}$
- Multivariate variable $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$
- Set of possible outcomes $\boldsymbol{x} \in \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$
- Each possible outcome is assigned a **probability value**

Marginal probability

- Marginal probabilities are probabilities defined for events corresponding to setting some components of the variable to given values
- Let $K \subseteq N$ and

$$\boldsymbol{X}_K := (X_k : k \in K)$$

$$\boldsymbol{x}_K \in \mathcal{X}_K := \bigtimes_{k \in K} \mathcal{X}_k$$

- Marginal probabilities follow from the additivity axiom:

$$P(\boldsymbol{X}_K = \boldsymbol{x}_K) = \sum_{\substack{\boldsymbol{z} \in \mathcal{X} \\ \boldsymbol{z}_K = \boldsymbol{x}_K}} p_{\boldsymbol{z}}$$

Marginal probability example

Problem setup

- $\boldsymbol{X} = (X_1, X_2, X_3)$
- $\mathcal{X}_1 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, $\mathcal{X}_2 = \{0, 1\}$, $\mathcal{X}_3 = \{+, -\}$
-

$\boldsymbol{x}$	x_1	A	A	A	A	B	B	B	B	C	C	C	C
	x_2	0	0	1	1	0	0	1	1	0	0	1	1
	x_3	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$p_{\boldsymbol{x}}$		0.2	0.1	0.1	0	0.1	0	0	0.3	0	0	0	0.2

Marginal probability example

Problem setup

- $\boldsymbol{X} = (X_1, X_2, X_3)$
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	x_1	A	A	A	A	B	B	B	B	C	C	C	C
$\boldsymbol{x}$	x_2	0	0	1	1	0	0	1	1	0	0	1	1
	x_3	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$p_{\boldsymbol{x}}$		0.2	0.1	0.1	0	0.1	0	0	0.3	0	0	0	0.2

Inferences

- $P(X_1 = \mathrm{B})$?
- $P(X_1 = \mathrm{A}, X_2 = 0)$?
- $P(X_1 = \mathrm{C}, X_3 = +)$?

Marginal probability example

Problem setup

- $\boldsymbol{X} = (X_1, X_2, X_3)$
- $\mathcal{X}_1 = \{\text{A}, \text{B}, \text{C}\}$, $\mathcal{X}_2 = \{0, 1\}$, $\mathcal{X}_3 = \{+, -\}$
-

	x_1	A	A	A	A	B	B	B	B	C	C	C	C
$\boldsymbol{x}$	x_2	0	0	1	1	0	0	1	1	0	0	1	1
	x_3	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$p_{\boldsymbol{x}}$		0.2	0.1	0.1	0	0.1	0	0	0.3	0	0	0	0.2

Inferences

- $P(X_1 = \text{B}) = 0.4$
- $P(X_1 = \text{A}, X_2 = 0) = 0.3$
- $P(X_1 = \text{C}, X_3 = +) = 0$

Conditional probabilities

- Conditional probabilities are probabilities that hold
  - assuming some event is known to be true, or specifically
  - assuming some random variables take some given values
- Let $B \subseteq \mathcal{X}$ for which $P(B) > 0$, then for $A \subseteq \mathcal{X}$ we have $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Let $K \subseteq N$ and

$$\boldsymbol{X}_K := (X_k : k \in K)$$
$$\boldsymbol{x}_K \in \mathcal{X}_K := \bigtimes_{k \in K} \mathcal{X}_k$$

with $P(\boldsymbol{X}_K = \boldsymbol{x}_K) > 0$ then

$$P(\boldsymbol{X}_{N \setminus K} = \boldsymbol{x}_{N \setminus K} | \boldsymbol{X}_K = \boldsymbol{x}_K) = \frac{P(\boldsymbol{X}_K = \boldsymbol{x}_K, \boldsymbol{X}_{N \setminus K} = \boldsymbol{x}_{N \setminus K})}{P(\boldsymbol{X}_K = \boldsymbol{x}_K)}$$

Conditional probability example

Problem setup

- $\boldsymbol{X} = (X_1, X_2, X_3)$
- $\mathcal{X}_1 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, $\mathcal{X}_2 = \{0, 1\}$, $\mathcal{X}_3 = \{+, -\}$
- Table:

	x_1	A	A	A	A	B	B	B	B	C	C	C	C
$\boldsymbol{x}$	x_2	0	0	1	1	0	0	1	1	0	0	1	1
	x_3	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$p_{\boldsymbol{x}}$		0.2	0.1	0.1	0	0.1	0	0	0.3	0	0	0	0.2

Conditional probability example

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	x_1	A	A	A	A	B	B	B	B	C	C	C	C
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	x_3	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$
$p_{\boldsymbol{x}}$		0.2	0.1	0.1	0	0.1	0	0	0.3	0	0	0	0.2

Inferences

- $P(X_1 = \mathrm{A} | X_2 = 0, X_3 = +)$?
- $P(X_1 = \mathrm{B} | X_2 = 0, X_3 = -)$?
- $P(X_2 = 1 | X_1 = \mathrm{C}, X_3 = +)$?

Conditional probability example

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$p_{\boldsymbol{x}}$		0.2	0.1	0.1	0	0.1	0	0	0.3	0	0	0	0.2

Inferences

- $P(X_1 = \mathrm{A} | X_2 = 0, X_3 = +) = \frac{2}{3}$
- $P(X_1 = \mathrm{B} | X_2 = 0, X_3 = -) = 0$
- $P(X_2 = 1 | X_1 = \mathrm{C}, X_3 = +)$ is not well-defined (*why?*)

Independence

- Two random variables X_1 and X_2 are independent if their well-defined conditionals coincide with the marginals for all $\boldsymbol{x} \in \mathcal{X}$:

$$
\begin{aligned}
P(X_1 = x_1 | X_2 = x_2) &= P(X_1 = x_1) \\
P(X_2 = x_2 | X_1 = x_1) &= P(X_2 = x_2)
\end{aligned}
$$

- This is equivalent to the joint factorizing:

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2) \text{ for all } \boldsymbol{x} \in \mathcal{X}$$

Reasoning — deducing inferences and making decisions

- What can be done with (joint) probabilities
  can also be done with marginal and conditional probabilities

Reasoning — deducing inferences and making decisions

- What can be done with (joint) probabilities
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Learning — creating a representation

Marginal and conditional probabilities can be either

- deduced from the learned (joint) probabilities, or
- learned directly, to together define the joint,
  possibly using independence assumptions

Multivariate probability — exercise

Consider the (joint) random variable $\boldsymbol{X} = (X_1, X_2, X_3)$ with $\mathcal{X} = \{0, 1\}^3$.
The joint probabilities are determined by the following information:

- X_1 and X_2 are independent
- It holds that $P(X_1 = 0) = 1/2$ and $P(X_2 = 0) = 1/5$
- The conditional probabilities $P(X_3 \mid X_1, X_2)$ are determined by the following table:

x_1	0	0	1	1
x_2	0	1	0	1
$P(X_3 = 0 \mid X_1 = x_1, X_2 = x_2)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

Derive that $P(X_1 = 0 \mid X_2 = 0, X_3 = 0) = \frac{5}{8}$.

As part of your calculation, you should derive and write a general expression for $P(X_1 \mid X_2, X_3)$ in terms of the (symbolic) probabilities that are given—before filling in the specific numbers.

Overview

A forecast states:

"There is an 80% probability of showers for tomorrow"

What does this mean?

(More generally, what is the meaning of probability values?)

Diversity in interpretations of probability

> There are several schools of thought regarding the interpretation of probabilities, none of them without flaws, internal contradictions, or paradoxes.
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> De Elía & Laprise (2005)

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Major interpretations

physical	*evidential*	*graded belief*
frequentist	classical	subjective
propensity	logical	

Physical

- Concerns statements about events related to physical systems
- Connected to frequency of occurrence of these events

Evidential

- A measure for the evidence supporting some (any) statement
- Typically intended to be objective

Graded belief

- A degree of belief about some (any) statement
- Typically subjective

Frequentist (physical interpretation)

An event's probability is defined as the **limit** of its **relative frequency** in many trials (the long-run probability).

Kaplan (2014)

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How short or long can or should the sequence of trials be?

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> Assuming **equally possible** cases, the probability of an event is the **ratio** of the relative number of cases favorable to it.
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> Wikipedia (paraphrased)

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What does the assumption mean and what are its grounds?

Subjective (graded belief interpretation)

A subjective probability is anyone's **opinion** of what the probability is for an event.

Siegel & Wagner (2022)

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Is it a problem if probability values are opinions?

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Is it a problem if probability values are opinions?

Betting interpretation (special case of subjective interpretation)

- Probabilities are defined by the subject's betting behavior (therefore also called *behavioral interpretation*)
- Relevant for our discussion of imprecise probability theories

Betting game setup

- Two players:
  - **subject** (gambler)
  - **bookie** (proposes bets)

- **Gambles** from the subject's perspective:

$$1_x(z) = \begin{cases} 1 & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases}$$

for all $x \in \mathcal{X}$

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Eliciting probabilities

- For each of the gambles 1_x, the subject offers their **fair price** p_x
- The bookie can propose to exchange one for the other, i.e., $1_x - p_x$ or $p_x - 1_x$, which the subject is committed to accept

- The p_x are the subject's probabilities

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- The p_x are the subject's probabilities
- Should the subject state any set of fair prices?

Coherence: deriving the axioms of probability from the betting game

- Assume the gambler specifies a negative price $p_x < 0$.
- Exchange proposed by the bookie:

$$p_x - 1_x = \begin{cases} p_x - 1 < 0 & \text{if } x \text{ occurs} \\ p_x < 0 & \text{otherwise} \end{cases}$$

- Should be unacceptable to the subject, because it implies a *sure loss*, i.e., is negative whatever occurs!
- **Nonnegativity axiom is required**

Coherence: deriving the axioms of probability from the betting game

- The bookie can choose to propose multiple bets, concerning some $S \subseteq \mathcal{X}$.
- Combined exchange proposed by the bookie:

$$\sum_{x \in S} (p_x - 1_x) = \Big(\sum_{x \in S} p_x\Big) - 1_S, \text{ with } 1_S = \sum_{x \in S} 1_x = \begin{cases} 1 & \text{if } S \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

- The fair price $P(S)$ for the combined bet is $\sum_{x \in S} p_x$
- **Additivity axiom follows**

Coherence: deriving the axioms of probability from the betting game

- The bookie can choose to combine the bets for all $x \in \mathcal{X}$, so for the gamble $1_{\mathcal{X}} = 1$
- Possible exchanges to be proposed: $P(\mathcal{X}) - 1$ or $1 - P(\mathcal{X})$
- Exchanges do not depend on the $x \in \mathcal{X}$ that occurs
- If $P(\mathcal{X}) \neq 1$, one possible proposal would result in a *sure loss*
- **The normedness axiom is required**

Overview

- Kick-off (slot 1)
- Classical probability theory (slot 1)
- Interpretation of probability (slot 2)
- Limitations of probability theory (slot 2)
  - Aleatoric vs. epistemic uncertainty
  - Representing epistemic uncertainty
  - Rational agents vs. Real agents
  - Why go beyond probability theory?
- Probability intervals (slot 3)
- Credal sets (slot 3–4)
- Acceptability & Desirability (slot 4–5)
- Interval expectation & probability (slot 5–6)

Aleatoric uncertainty — the irreducible part

- Aleatoric uncertainty arises from *random variation*
- Additional information cannot reduce it
- Other names: *variability*, *stochasticity*, *randomness*, *chance*, *risk*

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Epistemic uncertainty — the reducible part

- Epistemic uncertainty arises from *a lack of knowledge* (also at inference time)
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- Other names: *incertitude*, *ambiguity*, *ignorance*, *imprecision*

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- Other names: *variability*, *stochasticity*, *randomness*, *chance*, *risk*
- Sources: spatial variation, temporal fluctuations, manufacturing or genetic differences,. . .

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- Epistemic uncertainty arises from *a lack of knowledge* (also at inference time)
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- Other names: *incertitude*, *ambiguity*, *ignorance*, *imprecision*
- Sources: limited sample size, mensurational limits ('measurement error'), censoring, poorly defined outcomes,. . .

Aleatoric uncertainty examples

- Outcome of toss of fair coin

Epistemic uncertainty examples

- Bias of a coin for tossing

Aleatoric uncertainty examples

- Outcome of toss of fair coin
- Decay time of a radioactive atom

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Aleatoric uncertainty examples

- Outcome of toss of fair coin
- Decay time of a radioactive atom
- Value of a decimal of a randomly generated number
- 'Noise'

Epistemic uncertainty examples

- Bias of a coin for tossing
- Weight of a proton
- Value of a decimal of an irrational mathematical constant
- Model uncertainty
  - Parameter values
  - Dependencies
  - Functional forms
  - Level of abstraction

Can probability theory differentiate between aleatoric and epistemic uncertainty?

The importance of sample size: coin-flipping example

Two different coins

Coin	S	L
Flips	2	$2 \cdot 10^6$
Heads	50%	50%
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- How would you communicate derived inferences and decisions?

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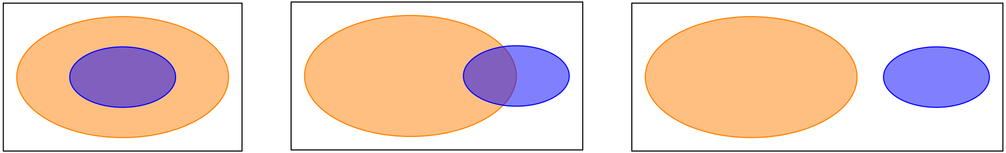
for both coins

- What can you say about the reliability of the estimate for each coin?
- How would you communicate the probability estimates?
- How would you communicate derived inferences and decisions?
- What about inferences depending on 1000 random variables with varying reliabilities?

Unknown dependence

- Nature of dependence between two events A and B often not known



Unknown dependence and Fréchet's bounds

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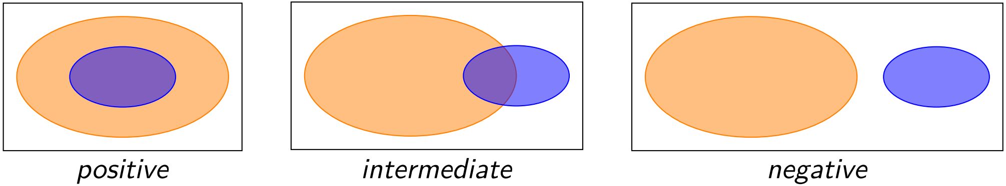

- Fréchet's bounds:

$$P(A \wedge B) \in \Big[\max\{0, P(A) + P(B) - 1\}, \min\{P(A), P(B)\}\Big]$$

$$P(A \vee B) \in \Big[\max\{P(A), P(B)\}, \min\{1, P(A) + P(B)\}\Big]$$

Can probability theory differentiate between aleatoric and epistemic uncertainty?

> [Aleatoric and epistemic uncertainty] must be treated *differently*;
>
> variability should be modeled as randomness
> with the methods of probability theory;
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> incertitude should be modeled as ignorance
> with the methods of interval analysis.
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Interval analysis

- Computation with intervals instead of with single numbers
- Example: $[2, 4] - [3, 5] = [-3, 1]$
- Ideal is to obtain tightest bounds
- In practice often outer bounds are used for computational reasons

Rational agents

In Savage's classical account of *Subjective Expected Utility Theory*, a **'rational' agent**

- models uncertainty in a problem **using a single probability measure**
- chooses between alternatives by **maximizing expected utility**

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Urn example

- Urn with 20 red, 10 black, and 10 white balls
  The rational agent uses a pmf with
  $p_\mathrm{R} = 1/2$ and $p_\mathrm{B} = p_\mathrm{W} = 1/4$

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Urn example

- Urn with 20 red, 10 black, and 10 white balls
  The rational agent uses a pmf with
  $p_R = 1/2$ and $p_B = p_W = 1/4$
- Consider three gambles whose payoff depends on the outcome of a random draw from the urn:

	R	B	W
f_{RB}	\$100	\$100	\$0
f_{RW}	\$100	\$0	\$100
f_{BW}	\$0	\$100	\$100

  The agent is *indifferent* between f_{RB} and f_{RW}, which they *strictly prefer* to f_{BW}

Are real agents irrational?

Risk aversion

- Urn with 20 red, 20 gray balls
- Two gambles:

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f_{RG}	\$50	\$50
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Ambiguity aversion

- Two urns; the agent must choose one:
  - E 20 black balls, 20 white balls
  - U unknown proportion of black and white balls
- One gamble:

	B	W
f_B	\$100	\$0

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  - E 20 black balls, 20 white balls
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- One gamble:

	B	W
f_B	\$100	\$0

- Rational agents must choose a pmf for U; in case they choose $p_B = p_W = 1/2$ they are *indifferent* between urns E and U
- Real agents *strictly prefer* urn E over urn U

Are real agents irrational?

Ellsberg paradox

- Urn with 20 red balls and 40 white or black balls in unknown proportion
- Four gambles:

	R	B	W
f_R	\$100	\$0	\$0
f_B	\$0	\$100	\$0
$f_{RW} = f_R + f_W$	\$100	\$0	\$100
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- Choose between f_R and f_B
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- Choose between f_R and f_B

  Real agents
  strictly prefer f_R over f_B,
  for a rational agent implying
  $p_R > p_B$

- Choose between f_{RW} and f_{BW}

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  strictly prefer f_{BW} over f_{RW},
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  $p_R < p_B$

Why go beyond probability theory?

To be able to deal with epistemic uncertainty:

- Distinguish sample size in uncertainty representation
- Express partial or missing knowledge

To let agents act according to a less restrictive definition of rationality:

- Be able to reflect justified aversions
- Model behavior that would otherwise be paradoxical

Overview

Kick-off (slot 1)

Classical probability theory (slot 1)

Interpretation of probability (slot 2)

Limitations of probability theory (slot 2)

Probability intervals (slot 3)
- Illustration: Blood groups
- Representation
- Reasoning

Credal sets (slot 3–4)

Acceptability & Desirability (slot 4–5)

Interval expectation & probability (slot 5–6)

Blood groups problem with cheap test

Problem setup

- A partial information sample:

A,B	AB	O	A,B
A,B	A,B	A,B	O

- A disease

- Two treatments with differing effectiveness:

	A	B	AB	O
f_1	0.5	0.6	0.7	0.1
f_2	0.4	0.3	0.3	0.8

Representation

PMF from observed frequencies:

p_A	p_B	p_{AB}	p_O
?	?	$\frac{1}{8}$	$\frac{2}{8}$

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Reasoning

- Bounds on $P(\{\text{A}, \text{O}\})$?
- Bounds on $E_p(f_1)$?
- Outcome with maximal lower probability?
- Treatment with highest upper expected effectiveness?

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$[0, \frac{5}{8}]$	$[0, \frac{5}{8}]$	$\frac{1}{8}$	$\frac{2}{8}$

Reasoning

- $P(\{A, O\}) \in [\frac{2}{8}, \frac{7}{8}]$
- $E_p(f_1) \in= [\frac{3.4}{8}, \frac{3.9}{8}] = [0.425, 0.4875]$
- Outcome O has maximal lower probability
- Both treatments have equal upper expected effectiveness

Basic setup, axioms, and terminology

Basic setup of **the theory of probability intervals** (Campos, Huete, Moral, 1994):

- Random variable X
- Finite set of outcomes $\mathcal{X}$
- Each outcome $x \in \mathcal{X}$ is assigned **lower and upper probability values** $(\underline{p}_x, \overline{p}_x)$

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A **probability interval** $(\underline{p}, \overline{p})$, a pair of lower and upper probability mass functions, must be:

1. **?**
2. **?**
3. **?**

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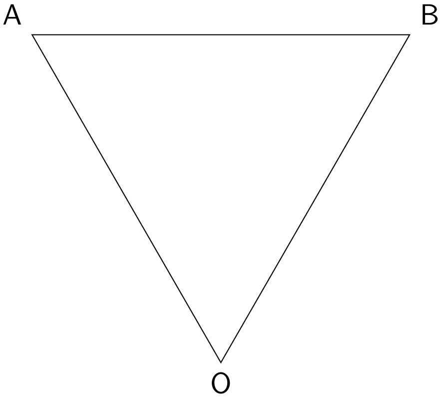
3. **?**

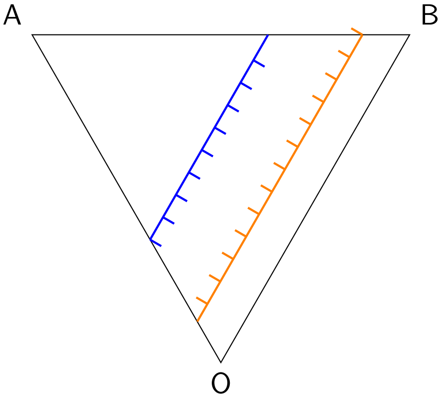
Unreachable bounds require tightening

	A	B	O
$\overline{p}$			
$\underline{p}$			



Unreachable bounds require tightening

	A	B	O
$\overline{p}$	3/8		
$\underline{p}$	1/8		



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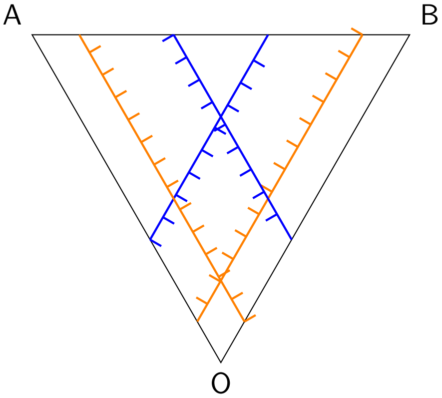
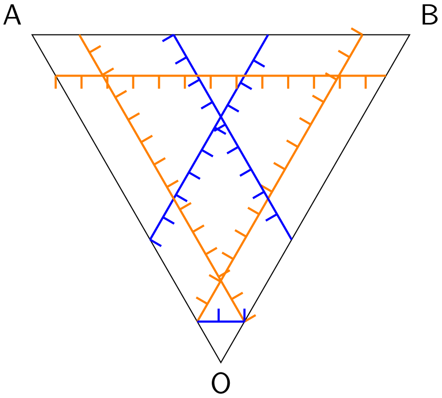


Unreachable bounds require tightening

	A	B	O
$\overline{p}$	$3/8$	$3/8$	$7/8$
$\underline{p}$	$1/8$	$1/8$	$1/8$



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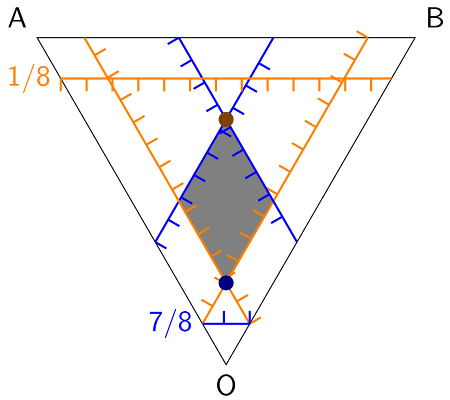

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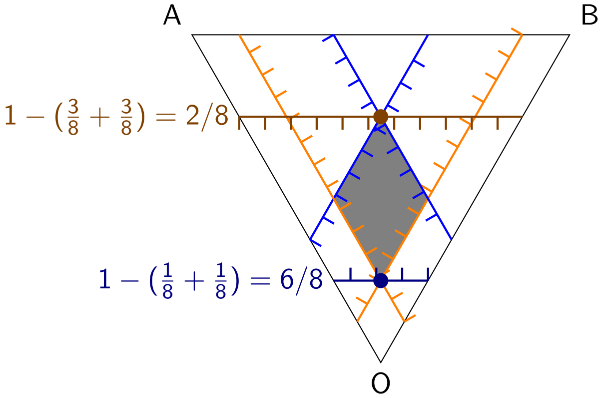

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3. Reachable: $\underline{p}_x \geq 1 - \sum_{z \neq x} \overline{p}_z$ and $\overline{p}_x \leq 1 - \sum_{z \neq x} \underline{p}_z$ for all outcomes $x \in \mathcal{X}$

Comparison by degree of imprecision

- A probability interval $(\underline{p}, \overline{p})$ is *included in* or *less imprecise than* a probability interval $(\underline{q}, \overline{q})$ if for all $x \in \mathcal{X}$

$$[\underline{p}_x, \overline{p}_x] \subseteq [\underline{q}_x, \overline{q}_x]$$

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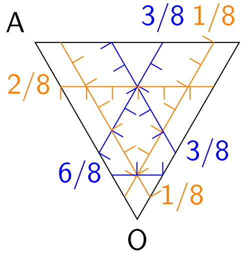
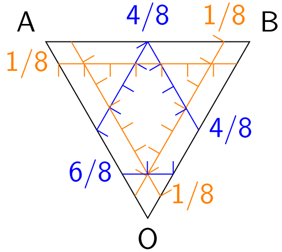


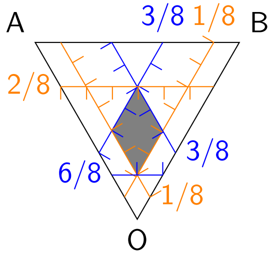


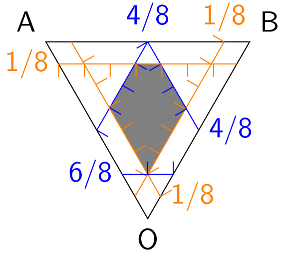
Comparison by degree of imprecision

- A probability interval $(\underline{p}, \overline{p})$ is *included in* or *less imprecise than* a probability interval $(\underline{q}, \overline{q})$ if for all $x \in \mathcal{X}$

$$[\underline{p}_x, \overline{p}_x] \subseteq [\underline{q}_x, \overline{q}_x]$$

- The corresponding credal sets will also respect the same inclusion relationship





Lower and upper probability mass sums

$$\underset{\sim}{P}(S) := \sum_{x \in S} \underline{p}_x, \qquad \widetilde{P}(S) := \sum_{x \in S} \overline{p}_x$$

Lower probability

$$\underline{P}(S) := \max\{\underset{\sim}{P}(S), 1 - \widetilde{P}(S^{\mathrm{c}})\}$$

Upper probability

$$\overline{P}(S) := \min\{\widetilde{P}(S), 1 - \underset{\sim}{P}(S^{\mathrm{c}})\}$$

Lower and upper probability mass sums

$$\underset{\sim}{P}(S) := \sum_{x \in S} \underline{p}_x, \qquad \widetilde{P}(S) := \sum_{x \in S} \overline{p}_x$$

	A	B	O	AB
$\overline{p}$	3/8	3/8	5/8	5/8
$\underline{p}$	1/8	1/8	3/8	0/8

Lower probability

$$\underline{P}(S) := \max\{\underset{\sim}{P}(S), 1 - \widetilde{P}(S^{\mathrm{c}})\}$$

Upper probability

$$\overline{P}(S) := \min\{\widetilde{P}(S), 1 - \underset{\sim}{P}(S^{\mathrm{c}})\}$$

Inferences

- Lower probability $\underline{P}(\{\mathrm{A}, \mathrm{B}\})$?
- Upper probability $\overline{P}(\{\mathrm{B}, \mathrm{O}\})$?

Lower and upper probability mass sums

$$\underset{\sim}{P}(S) := \sum_{x \in S} \underline{p}_x, \qquad \widetilde{P}(S) := \sum_{x \in S} \overline{p}_x$$

	A	B	O	AB
$\overline{p}$	3/8	3/8	5/8	5/8
$\underline{p}$	1/8	1/8	3/8	0/8

Lower probability

$$\underline{P}(S) := \max\{\underset{\sim}{P}(S), 1 - \widetilde{P}(S^c)\}$$

Upper probability

$$\overline{P}(S) := \min\{\widetilde{P}(S), 1 - \underset{\sim}{P}(S^c)\}$$

Inferences

- $\underline{P}(\{A, B\}) = 2/8$
- $\overline{P}(\{B, O\}) = 7/8$

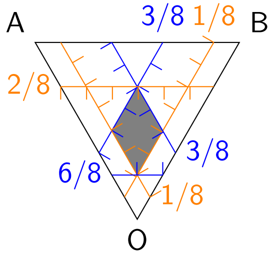
Comparison by degree of imprecision

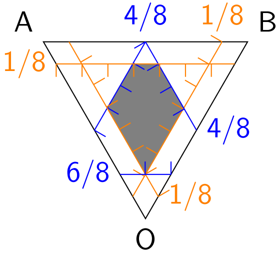
- A probability interval $(\underline{p}, \overline{p})$ is *included in* or *less imprecise than* a probability interval $(\underline{q}, \overline{q})$ if for all $x \in \mathcal{X}$

$$[\underline{p}_x, \overline{p}_x] \subseteq [\underline{q}_x, \overline{q}_x]$$

- The corresponding credal sets will also respect the same inclusion relationship
- Probability bounds will then respect the same inclusion relationship for all $S \subseteq \mathcal{X}$:

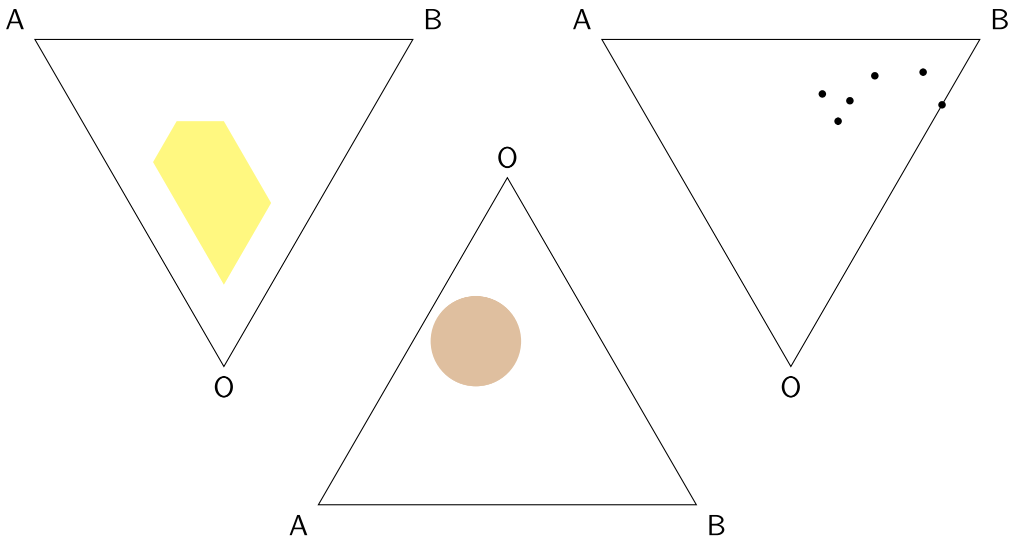
$$[\underline{P}(S), \overline{P}(S)] \subseteq [\underline{Q}(S), \overline{Q}(S)]$$





Overview

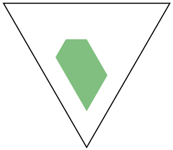
- Kick-off (slot 1)
- Classical probability theory (slot 1)
- Interpretation of probability (slot 2)
- Limitations of probability theory (slot 2)
- Probability intervals (slot 3)
- Credal sets (slot 3–4)
  - Representation
  - Reasoning
  - Multivariate credal sets
- Acceptability & Desirability (slot 4–5)
- Interval expectation & probability (slot 5–6)


A
B
O
O
A
B
A
B
O

Credal sets determined by bounds

Set of pmfs (probability measures) determined by *non-strict constraints* specified as

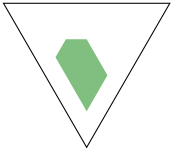
- a probability interval



Credal sets determined by bounds

Set of pmfs (probability measures) determined by *non-strict constraints* specified as

- a probability interval
- interval (lower & upper) probabilities or expectations (later)



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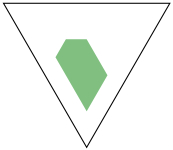
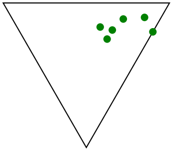


Directly specified credal sets

Set of pmfs (probability measures) specified *directly and explicitly* as such by

- discrete sets (‘*sets of Bayesians*’)



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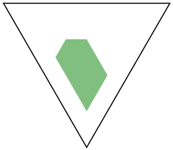
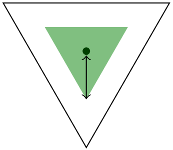


Directly specified credal sets

Set of pmfs (probability measures) specified *directly and explicitly* as such by

- discrete sets (‘*sets of Bayesians*’)
- neighborhoods of specific pmfs (‘*robust Bayesians*’)



Credal sets determined by bounds

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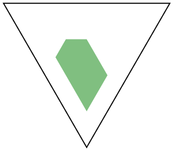


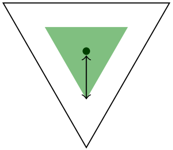
Convex and closed sets

Computationally convenient

Directly specified credal sets

Set of pmfs (probability measures) specified *directly and explicitly* as such by

- discrete sets ('*sets of Bayesians*')
- neighborhoods of specific pmfs ('*robust Bayesians*')



Generally non-convex and not closed

Can have impact on, e.g., decision rules

Basic setup, axioms, and terminology

Basic setup of **the theory of credal sets**:

- Random variable X
- Set of outcomes $\mathcal{X}$
- A *credal set* of probability mass functions for X is specified

Basic setup, axioms, and terminology

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- Uses representation from classical probability as building block

Basic setup, axioms, and terminology

Basic setup of **the theory of credal sets**:

- Random variable X
- Set of outcomes $\mathcal{X}$
- A *credal set* of probability mass functions for X is specified
- Uses representation from classical probability as building block

Axioms

A **credal set** $\mathcal{C}$ must be:

1. A subset of the set of all probability mass functions for X: $\mathcal{C} \subseteq \mathcal{P}_X$
2. Non-empty: $\mathcal{C} \neq \emptyset$

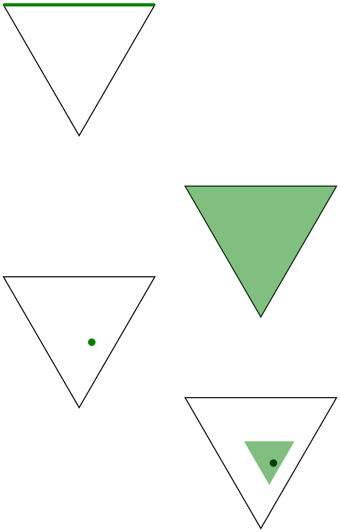
Special types of credal sets

- The vacuous credal set

$$\mathcal{P}_X^S := \{p \in \mathcal{P}_X : P(S) = 1\}$$

  relative to some event $S \subseteq \mathcal{X}$
  expresses that $X \in S$ and nothing more
- The vacuous credal set $\mathcal{P}_X^{\mathcal{X}} = \mathcal{P}_X$
  expresses complete ignorance
- Singleton credal sets $\{p\}$ correspond to
  the unique pmf p they contain
- Linear-vacuous or ε-contamination credal sets
  are a simple neighborhood model:

$$\mathcal{C}^{p,\varepsilon} := \{(1-\varepsilon)p + \varepsilon q : q \in \mathcal{P}_X\}$$



Lower & upper probability as lower & upper envelopes

$$\underline{P}(S) := \inf_{p \in \mathcal{C}} P_p(S) \qquad \overline{P}(S) := \sup_{p \in \mathcal{C}} P_p(S)$$

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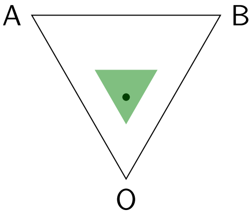
Envelope calculation as linear optimization over the credal set

$$\underline{E}(f) = \inf_{p \in \mathcal{C}} E_p(f) = \inf_{p \in \mathcal{C}} \sum_{x \in \mathcal{X}} p_x f(x)$$

For finite or closed credal sets $\mathcal{C}$, inf/sup becomes min/max

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
- pmf
  $p = (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{O}) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

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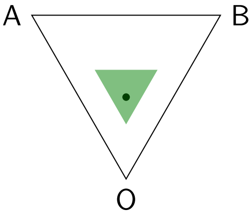

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Inferences

- Lower probability
- Lower expectation

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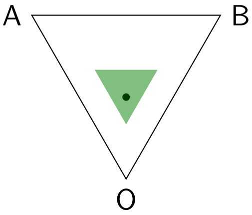

- function
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Inferences

- Lower probability

$\underline{P}(\{\mathrm{B}, \mathrm{O}\})$

- Lower expectation

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
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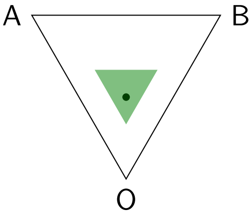


- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

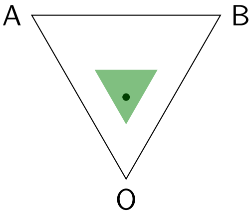
- Lower probability

$$\underline{P}(\{\mathrm{B}, \mathrm{O}\}) = \min_{q \in \mathcal{P}_X} \left(\frac{2}{3}(p_\mathrm{B} + p_\mathrm{O}) + \frac{1}{3}(q_\mathrm{B} + q_\mathrm{O})\right)$$

- Lower expectation

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
- pmf
  $p = (p_A, p_B, p_O) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(A), f(B), f(O)) = (1, 0, -1)$

Inferences

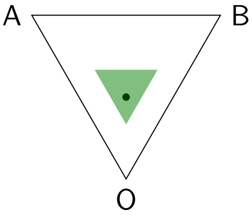
- Lower probability

$$\begin{aligned}\underline{P}(\{B, O\}) &= \min_{q \in \mathcal{P}_X} \Big(\tfrac{2}{3}(p_B + p_O) + \tfrac{1}{3}(q_B + q_O)\Big) \\ &= \tfrac{2}{3}\tfrac{3}{4} + \tfrac{1}{3}0 \\ &= \tfrac{1}{2}\end{aligned}$$

- Lower expectation

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
- pmf
  $p = (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{O}) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\underline{P}(\{\mathrm{B},\mathrm{O}\}) = \min_{q\in\mathcal{P}_X} \left(\frac{2}{3}(p_\mathrm{B} + p_\mathrm{O}) + \frac{1}{3}(q_\mathrm{B} + q_\mathrm{O})\right) = \frac{2}{3}\frac{3}{4} + \frac{1}{3}0 = \frac{1}{2}$$

- Lower expectation

$\underline{E}(f)$

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
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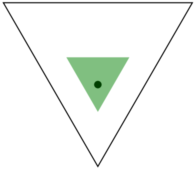
  $p = (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{O}) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\underline{P}(\{\mathrm{B},\mathrm{O}\}) = \min_{q\in\mathcal{P}_X}\Big(\frac{2}{3}(p_\mathrm{B}+p_\mathrm{O}) + \frac{1}{3}(q_\mathrm{B}+q_\mathrm{O})\Big) = \frac{2}{3}\frac{3}{4} + \frac{1}{3}0 = \frac{1}{2}$$

- Lower expectation

$$\underline{E}(f) = \min_{q\in\mathcal{P}_X}\Big(\frac{2}{3}\sum_{x\in\mathcal{X}} p_x f(x) + \frac{1}{3}\sum_{x\in\mathcal{X}} q_x f(x)\Big) = \min_{q\in\mathcal{P}_X}\Big(\frac{2}{3}(p_\mathrm{A}-p_\mathrm{O}) + \frac{1}{3}(q_\mathrm{A}-q_\mathrm{O})\Big)$$

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
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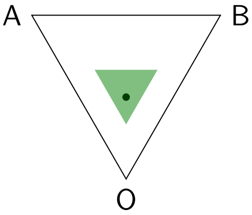

- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\begin{aligned}\underline{P}(\{\mathrm{B},\mathrm{O}\}) &= \min_{q\in\mathcal{P}_X}\Big(\tfrac{2}{3}(p_\mathrm{B}+p_\mathrm{O}) + \tfrac{1}{3}(q_\mathrm{B}+q_\mathrm{O})\Big) \\ &= \tfrac{2}{3}\tfrac{3}{4} + \tfrac{1}{3}0 \\ &= \tfrac{1}{2}\end{aligned}$$

- Lower expectation

$$\begin{aligned}\underline{E}(f) &= \min_{q\in\mathcal{P}_X}\Big(\tfrac{2}{3}\sum_{x\in\mathcal{X}} p_x f(x) + \tfrac{1}{3}\sum_{x\in\mathcal{X}} q_x f(x)\Big) \\ &= \min_{q\in\mathcal{P}_X}\Big(\tfrac{2}{3}(p_\mathrm{A}-p_\mathrm{O}) + \tfrac{1}{3}(q_\mathrm{A}-q_\mathrm{O})\Big) \\ &= \tfrac{2}{3}(-\tfrac{1}{4}) + \tfrac{1}{3}(-1) \\ &= -\tfrac{1}{2}\end{aligned}$$

Lower & upper probability as lower & upper envelopes

$$\underline{P}(S) := \inf_{p \in \mathcal{C}} P_p(S) \qquad \overline{P}(S) := \sup_{p \in \mathcal{C}} P_p(S)$$

Lower & upper expectation as lower & upper envelopes

$$\underline{E}(f) := \inf_{p \in \mathcal{C}} E_p(f) \qquad \overline{E}(f) := \sup_{p \in \mathcal{C}} E_p(f)$$

Envelope calculation as linear optimization over the credal *set's extreme points*

$$\underline{E}(f) = \inf_{p \in \mathcal{C}} E_p(f) = \inf_{p \in \mathcal{C}} \sum_{x \in \mathcal{X}} p_x f(x) = \inf_{p \in \operatorname{ext} \mathcal{C}} \sum_{x \in \mathcal{X}} p_x f(x)$$

(Extreme points exist for closed credal sets and fully characterize convex ones.)

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
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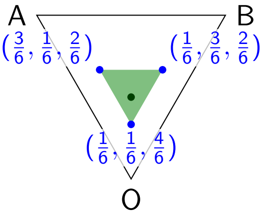
  $p = (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{O}) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability
- Lower expectation

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
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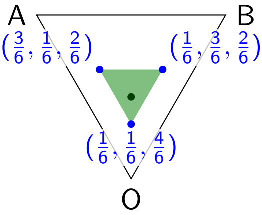

- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\underline{P}(\{\mathrm{B}, \mathrm{O}\})$$

- Lower expectation

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
- pmf
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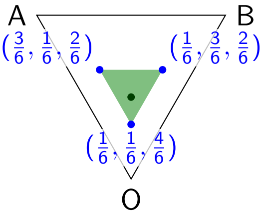

- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\underline{P}(\{\mathrm{B},\mathrm{O}\}) = \min_{q \in \mathrm{ext}\,\mathcal{C}^{p,\varepsilon}} (q_\mathrm{B} + q_\mathrm{O})$$

- Lower expectation

Example setup

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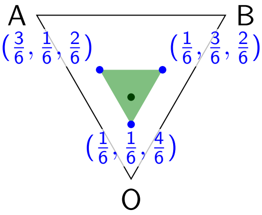

- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\underline{P}(\{\mathrm{B},\mathrm{O}\}) = \min_{q \in \mathrm{ext}\,\mathcal{C}^{p,\varepsilon}} (q_\mathrm{B} + q_\mathrm{O}) = \min\{\tfrac{3}{6}, \tfrac{5}{6}, \tfrac{5}{6}\} = \tfrac{3}{6} = \tfrac{1}{2}$$

- Lower expectation

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
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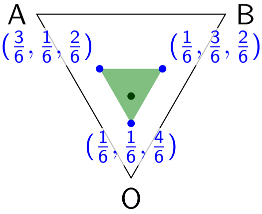
- pmf
  $p = (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{O}) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

Inferences

- Lower probability

$$\underline{P}(\{\mathrm{B}, \mathrm{O}\}) = \min_{q \in \mathrm{ext}\,\mathcal{C}^{p,\varepsilon}} (q_\mathrm{B} + q_\mathrm{O}) = \min\{\tfrac{3}{6}, \tfrac{5}{6}, \tfrac{5}{6}\} = \tfrac{3}{6} = \tfrac{1}{2}$$

- Lower expectation

$$\underline{E}(f)$$

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
- pmf
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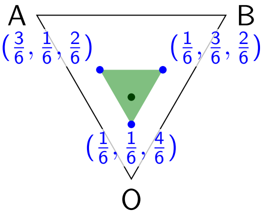

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- Lower expectation

$$\underline{E}(f) = \min_{q \in \operatorname{ext} \mathcal{C}^{p,\varepsilon}} \sum_{x \in \mathcal{X}} q_x f(x) = \min_{q \in \operatorname{ext} \mathcal{C}^{p,\varepsilon}} (q_\mathrm{A} - q_\mathrm{O})$$

Example setup

- linear-vacuous credal set $\mathcal{C}^{p,\varepsilon}$
- mixture coefficient $\varepsilon = \frac{1}{3}$
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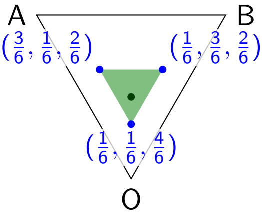
  $p = (p_\mathrm{A}, p_\mathrm{B}, p_\mathrm{O}) = (\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$



- function
  $f = (f(\mathrm{A}), f(\mathrm{B}), f(\mathrm{O})) = (1, 0, -1)$

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$$\underline{E}(f) = \min_{q \in \mathrm{ext}\, \mathcal{C}^{p,\varepsilon}} \sum_{x \in \mathcal{X}} q_x f(x) = \min_{q \in \mathrm{ext}\, \mathcal{C}^{p,\varepsilon}} (q_\mathrm{A} - q_\mathrm{O}) = \min\{\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{3}{6}\} = -\tfrac{3}{6} = -\tfrac{1}{2}$$

Multivariate credal sets: basic setup & basic idea

- Index set $N = \{1, \ldots, n\}$
- Multivariate variable $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$
- Set of possible outcomes $\boldsymbol{x} \in \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$
- A **joint credal set** $\mathcal{C}^{\boldsymbol{X}}$ of joint probability mass functions for $\boldsymbol{X}$ is specified

Apply probabilistic operations pointwise
to the elements of the credal sets involved

Marginal credal sets

- A marginal credal set is defined for a subset of the random variables
- Let $K \subseteq N$, then
  - $\boldsymbol{X}_K := (X_k : k \in K)$ and $\boldsymbol{x}_K \in \mathcal{X}_K := \times_{k \in K} \mathcal{X}_k$
  - Notation: $p^{\boldsymbol{X}_K}$ is the $\boldsymbol{X}_K$-marginal of the joint pmf $p^{\boldsymbol{X}}$

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  - Notation: $p^{\boldsymbol{X}_K}$ is the $\boldsymbol{X}_K$-marginal of the joint pmf $p^{\boldsymbol{X}}$
- A **marginal credal set** is obtained by marginalizing each of its member pmfs:

$$\mathcal{C}^{\boldsymbol{X}_K} = \left\{ p^{\boldsymbol{X}_K} : p^{\boldsymbol{X}} \in \mathcal{C}^{\boldsymbol{X}} \right\}$$

Marginal credal sets example

Problem setup

- Two random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
- Joint credal set $\mathcal{C}^{\boldsymbol{X}} := \mathcal{C}^{p,\varepsilon}$ with ε unspecified and p given below (in black), together with its marginals (in green)

p	0	1	p^{X_2}
$-$	$3/9$	$1/9$	$4/9$
$+$	$2/9$	$3/9$	$5/9$
p^{X_1}	$5/9$	$4/9$	

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Inference

Marginal credal set $\mathcal{C}^{X_1}$

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Inference

Marginal credal set $\mathcal{C}^{X_1}$

Solution

- Marginalize element $q = (1-\varepsilon)p + \varepsilon r$ of $\mathcal{C}^{\boldsymbol{X}}$, where $r \in \mathcal{P}_{\boldsymbol{X}}$:

$$q^{X_1}_{x_1} = \sum_{x_2 \in \mathcal{X}_2} q_{(x_1,x_2)} = (1-\varepsilon)p^{X_1}_{x_1} + \varepsilon r^{X_1}_{x_1}$$

Marginal credal sets example

Problem setup

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- Gather all marginalized elements:

$$\mathcal{C}^{X_1} = \mathcal{C}^{p^{X_1},\varepsilon}$$

because r^{X_1} ranges over $\mathcal{P}_{X_1}$ as r ranges over $\mathcal{P}_{\boldsymbol{X}}$

Marginal credal sets example

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q	0	1
$-$	$(1-\varepsilon)3/9 + \varepsilon r_{(0,-)}$	$(1-\varepsilon)1/9 + \varepsilon r_{(1,-)}$
$+$	$(1-\varepsilon)2/9 + \varepsilon r_{(0,+)}$	$(1-\varepsilon)3/9 + \varepsilon r_{(1,+)}$
q^{X_1}	$(1-\varepsilon)5/9 + \varepsilon r^{X_1}_0$	$(1-\varepsilon)4/9 + \varepsilon r^{X_1}_1$

Conditional credal sets

- A conditional credal set is determined by
  - assuming some event is known to be true, or specifically
  - assuming some random variables take some given values: $\boldsymbol{X}_K = \boldsymbol{x}_K$, with $K \subset N$
- Notation: $p^{\boldsymbol{X}_{N \setminus K} | \boldsymbol{x}_K}$ is the $\boldsymbol{X}_K$-conditional of the joint pmf $p^{\boldsymbol{X}}$

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$$\mathcal{C}^{\boldsymbol{X}_{N\setminus K}|\boldsymbol{x}_K} = \left\{ p^{\boldsymbol{X}_{N\setminus K}|\boldsymbol{x}_K} : \left(p^{\boldsymbol{X}} \in \mathcal{C}^{\boldsymbol{X}} \wedge p^{\boldsymbol{X}_K}(\boldsymbol{x}_K) > 0 \right) \right\}$$

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This is called *conditioning by regular extension*

Conditional credal sets example

Problem setup

- Two random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
- Joint credal set $\mathcal{C}^{\boldsymbol{X}} := \mathcal{C}^{\Gamma}$ with elements q given below (in black), together with its marginals (in green), with $\gamma \in \Gamma = [-\frac{3}{9}, \frac{3}{9}]$

q	0	1	q^{X_2}
$-$	$3/9 - \gamma$	$1/9$	$4/9 - \gamma$
$+$	$2/9$	$3/9 + \gamma$	$5/9 + \gamma$
q^{X_1}	$5/9 - \gamma$	$4/9 + \gamma$	

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Conditional credal set $\mathcal{C}^{X_1|+}$

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q^{X_1}	$5/9 - \gamma$	$4/9 + \gamma$	

Inference

Conditional credal set $\mathcal{C}^{X_1|+}$

Solution

- Condition elements q of $\mathcal{C}^{\boldsymbol{X}}$:

$$q^{X_1|+} = \frac{q_{(X_1,+)}}{q^{X_2}_{+}} = \frac{(2/9, 3/9 + \gamma)}{5/9 + \gamma} = (r, 1 - r),$$

with $r = \frac{2/9}{5/9+\gamma}$

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with $r = \frac{2/9}{5/9+\gamma}$

- Gather all conditioned elements:

$$\mathcal{C}^{X_1|+} = \left\{(r, 1 - r) : r \in [\tfrac{1}{4}, 1]\right\}$$

Conditional credal sets example

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(Exercise: show that $\mathcal{C}^{X_1|+} = \mathcal{C}^{(1,0),\frac{3}{4}}$)

Complete independence for credal sets

- Different generalizations of (stochastic) independence to credal sets are possible
- Here, we only consider the independence concept
  associated to the 'sets of Bayesians' interpretation
- Notation: $p \otimes q$ denotes the independent product of two pmfs p and q

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Definition of complete independence

Consider $\boldsymbol{X} = (X_1, X_2)$; the random variables X_1 and X_2 are *completely independent* if they are stochastically independent for each of the pmfs in the joint credal set $\mathcal{C}^{\boldsymbol{X}}$:

$$\mathcal{C}^{\boldsymbol{X}} \subseteq \mathcal{C}^{X_1} \otimes \mathcal{C}^{X_2} := \left\{ p^{X_1} \otimes p^{X_2} : p^{X_1} \in \mathcal{C}^{X_1}, p^{X_2} \in \mathcal{C}^{X_2} \right\}$$

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Not convex even if $\mathcal{C}^{X_1}$ and $\mathcal{C}^{X_2}$ are!

Complete independence example

Problem setup

- Random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
- Marginal credal sets $\mathcal{C}^{X_1} := \mathcal{C}^{(\frac{1}{2}, \frac{1}{2}), \frac{1}{2}}$ and $\mathcal{C}^{X_2} := \mathcal{C}^{(\frac{1}{3}, \frac{2}{3}), \frac{1}{3}}$

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Goal

Show that their completely independent joint credal set is not convex

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Show that their completely independent joint credal set is not convex

Demonstration

- We construct a counterexample to convexity, i.e., we construct a convex mixture of elements of the joint that lies outside the joint because it does not factorize

Complete independence example

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Show that their completely independent joint credal set is not convex

Demonstration

- We construct a counterexample to convexity, i.e., we construct a convex mixture of elements of the joint that lies outside the joint because it does not factorize
- An element q of $\mathcal{C}^{\boldsymbol{X}} = \mathcal{C}^{X_1} \otimes \mathcal{C}^{X_2}$ is defined for every $r^{X_1} \in \mathcal{P}_{X_1}$ and $r^{X_2} \in \mathcal{P}_{X_2}$ by

$$q_{(x_1,x_2)} = (\tfrac{1}{2} p^{X_1}_{x_1} + \tfrac{1}{2} r^{X_1}_{x_1})(\tfrac{2}{3} p^{X_2}_{x_2} + \tfrac{1}{3} r^{X_2}_{x_2})$$

Complete independence example

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- Random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
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Goal

Show that their completely independent joint credal set is not convex

Demonstration

$r_0^{X_1} = r_-^{X_2} = 0$

	0	1	q^{X_2}
$-$			$2/9$
$+$			$7/9$
q^{X_1}	$1/4$	$3/4$	

$r_0^{X_1} = r_-^{X_2} = 1$

	0	1	q^{X_2}
$-$			$5/9$
$+$			$4/9$
q^{X_1}	$3/4$	$1/4$	

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Goal

Show that their completely independent joint credal set is not convex

Demonstration

$r_0^{X_1} = r_-^{X_2} = 0$

	0	1	q^{X_2}
$-$	2/36	6/36	2/9
$+$	7/36	21/36	7/9
q^{X_1}	1/4	3/4	

$r_0^{X_1} = r_-^{X_2} = 1$

	0	1	q^{X_2}
$-$	15/36	5/36	5/9
$+$	12/36	4/36	4/9
q^{X_1}	3/4	1/4	

Complete independence example

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Goal

Show that their completely independent joint credal set is not convex

Demonstration

$r_0^{X_1} = r_-^{X_2} = 0$

$\frac{1}{2}$

	0	1	q^{X_2}
$-$	$2/36$	$6/36$	$2/9$
$+$	$7/36$	$21/36$	$7/9$
q^{X_1}	$1/4$	$3/4$	

$+\ \frac{1}{2}$

$r_0^{X_1} = r_-^{X_2} = 1$

	0	1	q^{X_2}
$-$	$15/36$	$5/36$	$5/9$
$+$	$12/36$	$4/36$	$4/9$
q^{X_1}	$3/4$	$1/4$	

$=$

	0	1	q^{X_2}
$-$	$17/72$	$11/72$	
$+$	$19/72$	$25/72$	
q^{X_1}			

Complete independence example

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$r_0^{X_1} = r_-^{X_2} = 0$

$\frac{1}{2}$

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$=$

not factorizing

	0	1	q^{X_2}
$-$	$17/72$	$11/72$	$7/18$
$+$	$19/72$	$25/72$	$11/18$
q^{X_1}	$1/2$	$1/2$	

Overview

Kick-off (slot 1)

Classical probability theory (slot 1)

Interpretation of probability (slot 2)

Limitations of probability theory (slot 2)

Probability intervals (slot 3)

Credal sets (slot 3–4)

Acceptability & Desirability (slot 4–5)

- Representation
- Reasoning
- Axiom variants
- Multivariate acceptability

Interval expectation & probability (slot 5–6)

Gambles

Earlier appearance of 'gamble':

- In the betting game, indicators 1_x and 1_S
- In urn problems, scaled indicators $f_i \propto 1_S$
- In blood groups example, effectiveness descriptions f_1, f_2, positive functions on $\mathcal{X}$

Gambles

Earlier appearance of ‘gamble’:

- In the betting game, indicators 1_x and 1_S
- In urn problems, scaled indicators $f_i \propto 1_S$
- In blood groups example, effectiveness descriptions f_1, f_2, positive functions on $\mathcal{X}$

Exchanges

- In the betting game, differences between fair prices and gambles $p_x - 1_x = p_x \cdot 1_{\mathcal{X}} - 1_x$ and $P(S) - 1_S$
- In urn problems, preferences between gambles f_i and f_j can be seen as preference between their difference $f_i - f_j$ and the zero gamble $0 = 0 \cdot 1_{\mathcal{X}}$ (status quo)
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Linear space of gambles

(Bounded) real-valued functions in $\mathcal{L} = (\mathcal{X} \to \mathbb{R})$, as foundation for representation
(or linear subspace of functions instead)

Basic setup, axioms, and terminology

Basic setup of **acceptability**:

- Random variable X
- Finite set of outcomes $\mathcal{X}$
- Each gamble $f \in \mathcal{L}$ is acceptable $(f \in \mathcal{D})$ or not $(f \notin \mathcal{D})$

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Axioms

A **set of acceptable gambles** $\mathcal{D}$ is **coherent** when:

D1. if $f \lessdot 0$, then $f \notin \mathcal{D}$ (*avoiding sure loss*)

D2. if $f \geq 0$, then $f \in \mathcal{D}$ (*accepting nonnegative gain*)

D3. if $f \in \mathcal{D}$ and $\lambda \in \mathbb{R}_{>}$, then $\lambda \cdot f \in \mathcal{D}$ (*positive scaling*)

D4. if $f, g \in \mathcal{D}$, then $f + g \in \mathcal{D}$ (*combination*)

Gamble (vector) inequalities: $f \gtrdot 0$ iff $\min f > 0$, $f \geq 0$ iff $\min f \geq 0$, $f > 0$ iff $f \geq 0$ and $f \neq 0$

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D1. $\mathcal{L}_{\lessdot} \cap \mathcal{D} = \emptyset$ (*avoiding sure loss*)

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D3. $\mathbb{R}_{>} \cdot \mathcal{D} \subseteq \mathcal{D}$ (*positive scaling*)

D4. $\mathcal{D} + \mathcal{D} \subseteq \mathcal{D}$ (*combination*)

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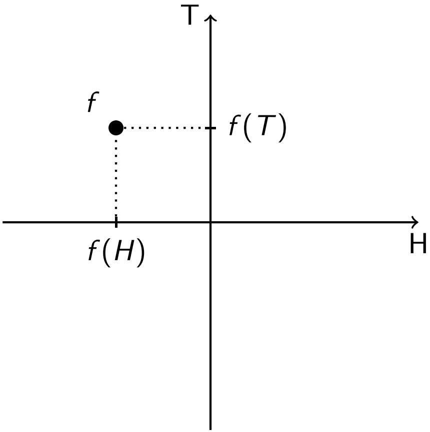
Visualizing gambles



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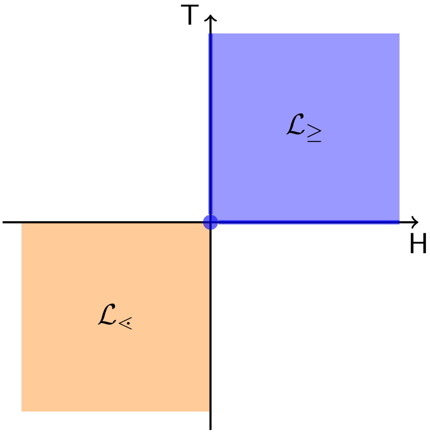

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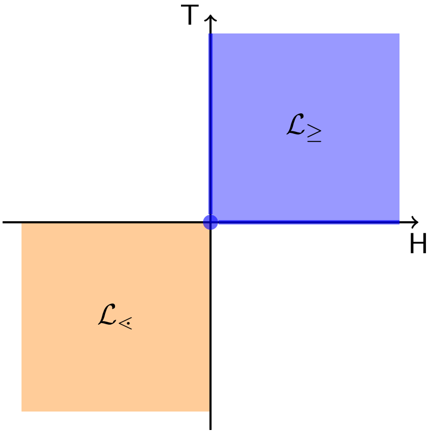
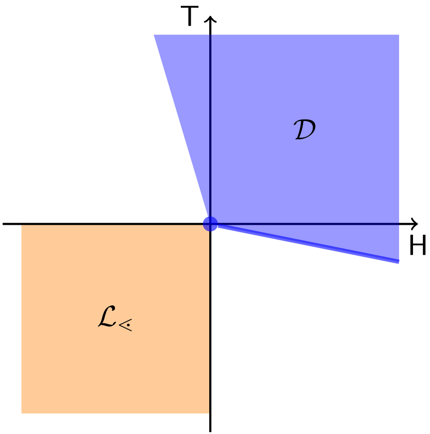


Coherent sets $\mathcal{D}$ are *cones* because of positive scaling and combination



From acceptability to preference

- Consider two gambles, f and g
- Depending on where their differences $f - g$ and $g - f$ lie, different preference relations hold between them

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Background preference/order implied by D1 and D2

- Non-strict preference: $f \succcurlyeq g$ iff $f \geq g$
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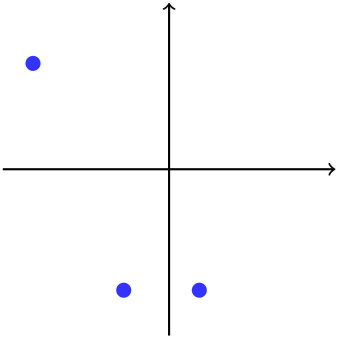

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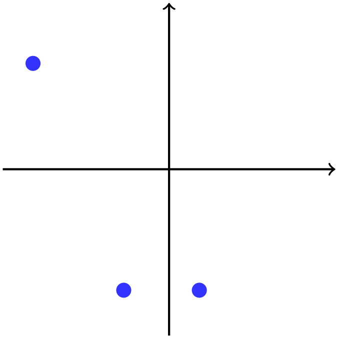

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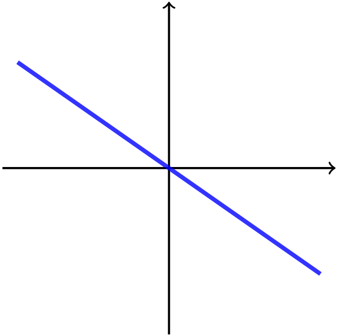

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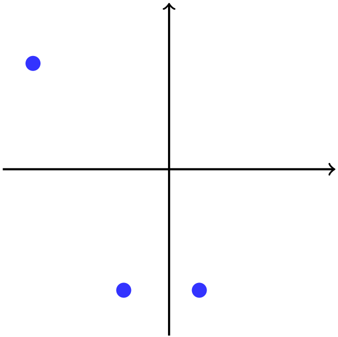

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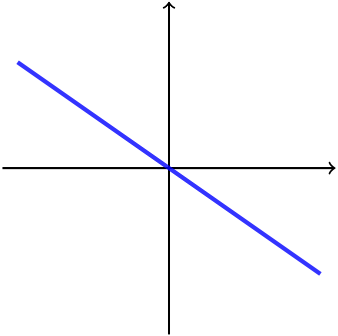

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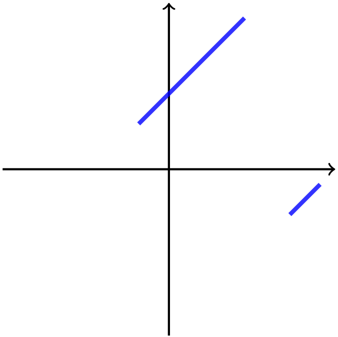

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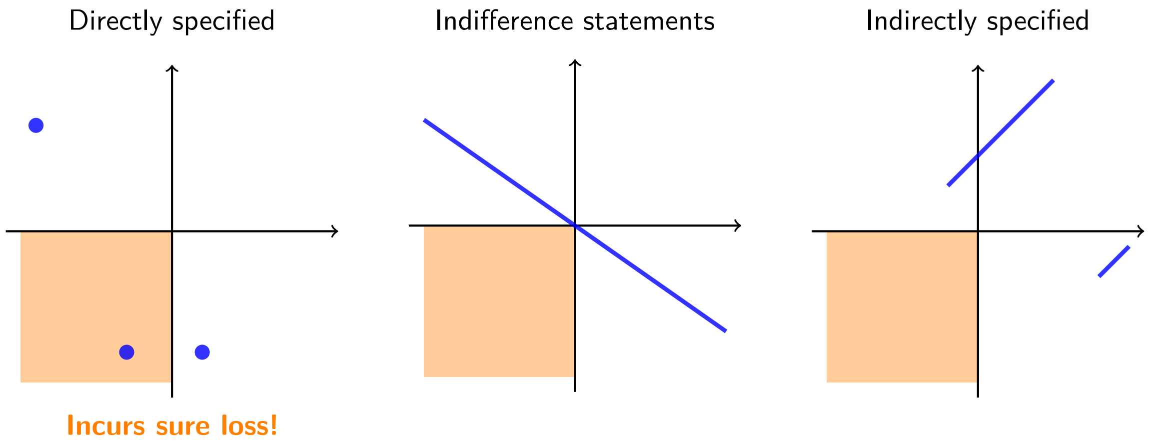

Positive linear hull operation

Two of the axioms are 'generative':

- Positive scaling (D3): if $f \in \mathcal{A}$, then all gambles in $\{\lambda \cdot f : \lambda \in \mathbb{R}_{>}\}$ must also be considered acceptable
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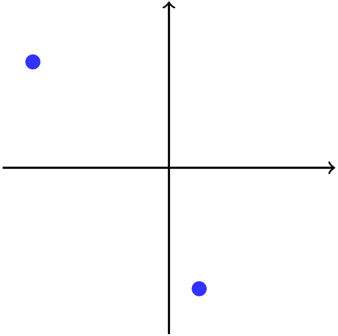

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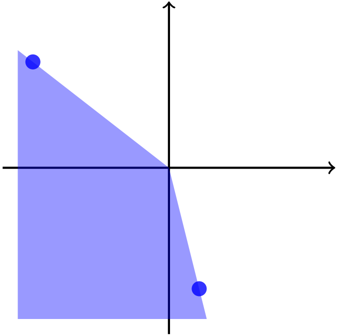

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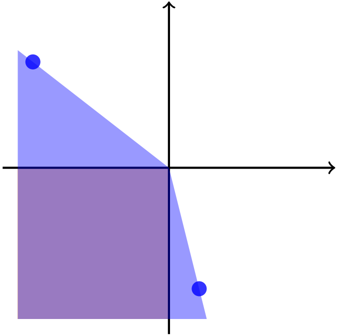

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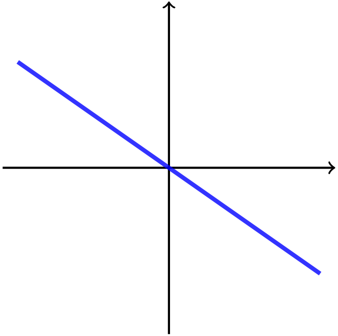

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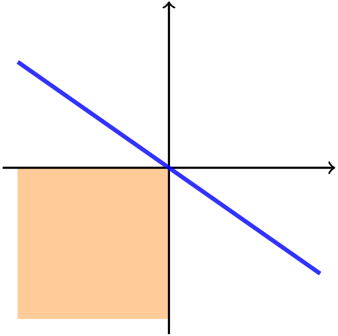

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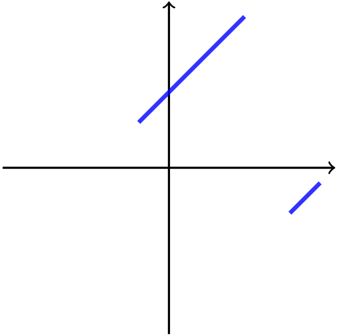

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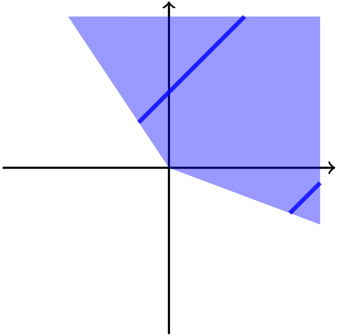

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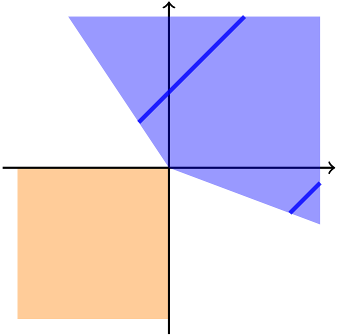

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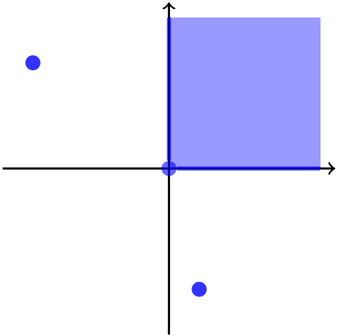

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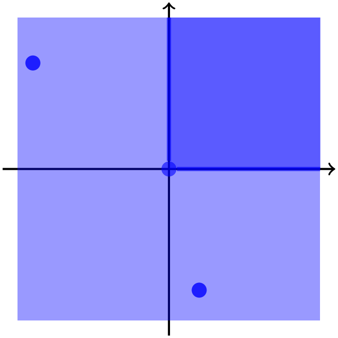

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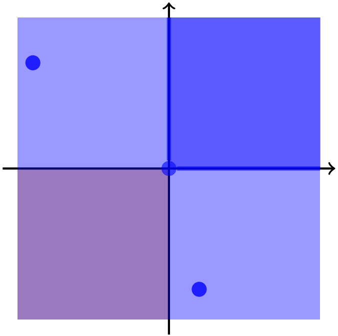

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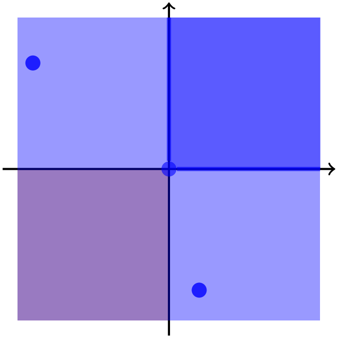

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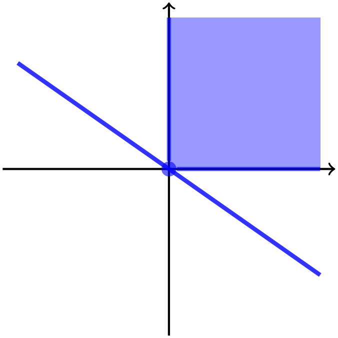

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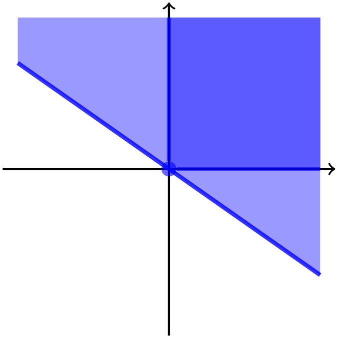

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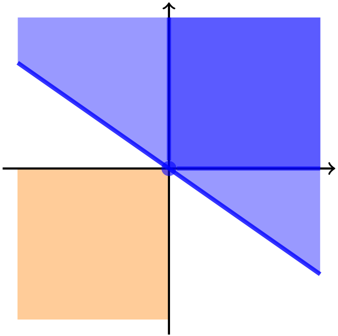

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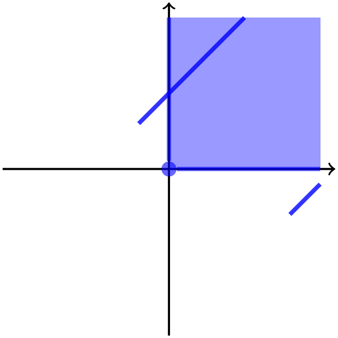

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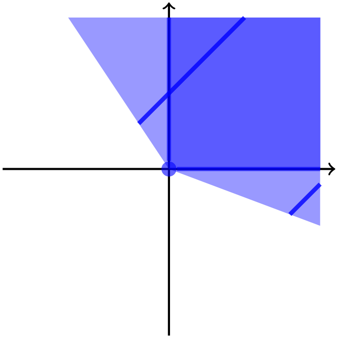

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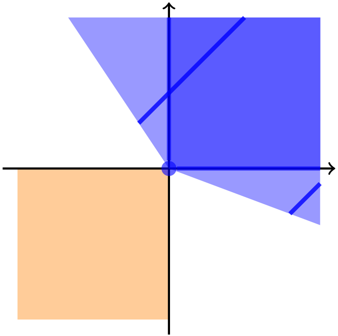

Least committal extension

Set of all coherent sets of acceptable gambles

- This set can be (partially) ordered according to set inclusion $\subset$
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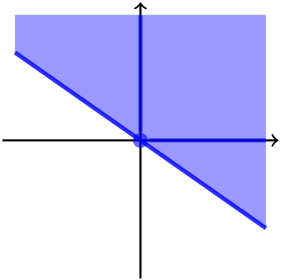

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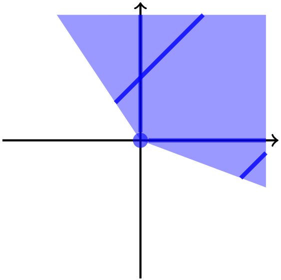

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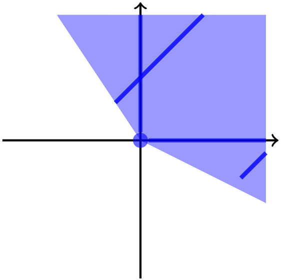

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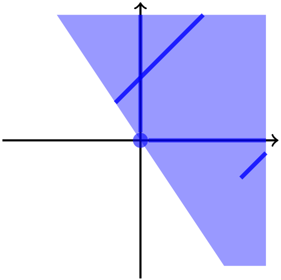

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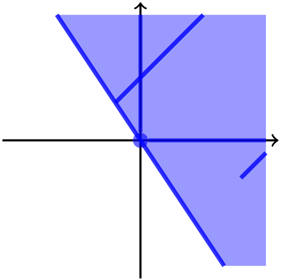

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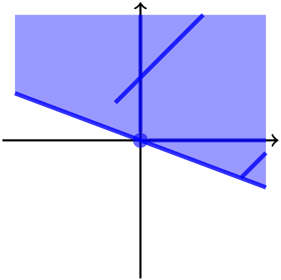

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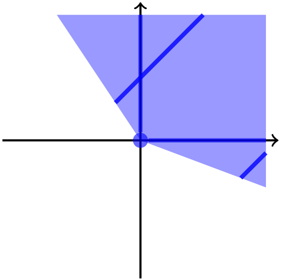

Coherent extensions of an assessment $\mathcal{A}$ that avoids sure loss

- In general, there are multiple coherent extensions of $\mathcal{A}$
- The **least committal extension** is the smallest one: it adds the least commitments (it is *conservative* in that regard)
- The least committal extension is equal to the intersection of all coherent extensions
- *The least committal extension coincides with the natural extension*

Changing how $\mathcal{D}$ is constrained

- Replacing the background sets $\mathcal{L}_{\lessdot}/\mathcal{L}_{\geq}$ in Axiom D1/D2 (extra relevant for infinite $\mathcal{X}$)
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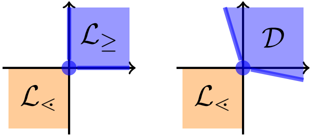
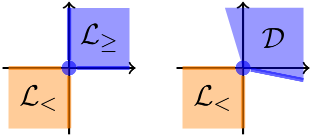
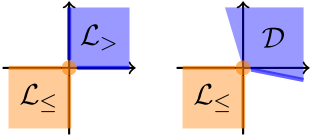
- $\mathcal{L}_{\lessdot}/\mathcal{L}_{\geq}$ and $\mathcal{D}$ closed (*almost desirability*, Walley, 1991)



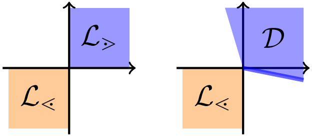
- $\mathcal{L}_{<}/\mathcal{L}_{\geq}$ (*favorable gambles*, Seidenfeld et al., 1990; Walley, 2000)



- $\mathcal{L}_{\leq}/\mathcal{L}_{>}$ (*desirability*, De Cooman, Thursday! D1 becomes $0 \notin \mathcal{D}$)



- $\mathcal{L}_{\lessdot}/\mathcal{L}_{\gtrdot}$ (*acceptable bets*, Williams, 1974)



Changing what shape $\mathcal{D}$ can take

- Replacing the generating rules in Axioms D3 and D4
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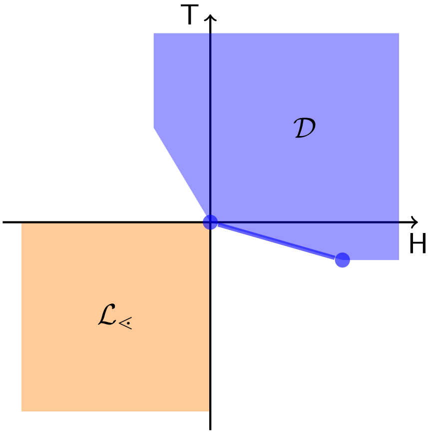
- *Example*: replace positive scaling and combination by *convexity*,

  if $f, g \in \mathcal{D}$ and $\mu \in [0, 1]$,
  then $\mu \cdot f + (1 - \mu) \cdot g \in \mathcal{D}$

  This means replacing the *positive linear hull* posi by the *convex hull* co



Multivariate acceptability: basic setup

- Index set $N = \{1, \dots, n\}$
- Multivariate variable $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$
- Set of possible outcomes $\boldsymbol{x} \in \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$
- Linear space of gambles $\mathcal{L}$ on $\mathcal{X}$
- We assume given a coherent set of acceptable gambles $\mathcal{D} \subset \mathcal{L}$

Marginal set of acceptable gambles

- A marginal set of acceptable gambles is defined on a linear subspace of $\mathcal{L}$
- We consider subspaces defined by $\mathcal{L}_K := (\mathcal{X}_K \to \mathbb{R})$, with $K \subseteq N$
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- For this, we lift it to its **cylindrical extension** $\uparrow_{\boldsymbol{x}_{N\setminus K}} f$ defined by

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- The marginal set of acceptable gambles is the inverse image of the cylindrical extension:

$$\mathcal{D}_K := \uparrow^{-1}_{\boldsymbol{x}_{N\setminus K}} \mathcal{D} = \left\{ f \in \mathcal{L}_K : \uparrow_{\boldsymbol{x}_{N\setminus K}} f \in \mathcal{D} \right\}$$

Conditional set of acceptable gambles

- A conditional set of acceptable gambles is determined by
  - assuming some event is known to be true, or specifically
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Epistemic irrelevance & independence

- We here consider the case $|N| = 2$
- The random variable X_2 is **epistemically irrelevant** to X_1 if the X_2-conditionals coincide with the X_1-marginal for all $x_2 \in \mathcal{X}_2$:

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- *Structural assumptions* such as these can be combined with natural extension

Overview

Kick-off (slot 1)

Classical probability theory (slot 1)

Interpretation of probability (slot 2)

Limitations of probability theory (slot 2)

Probability intervals (slot 3)

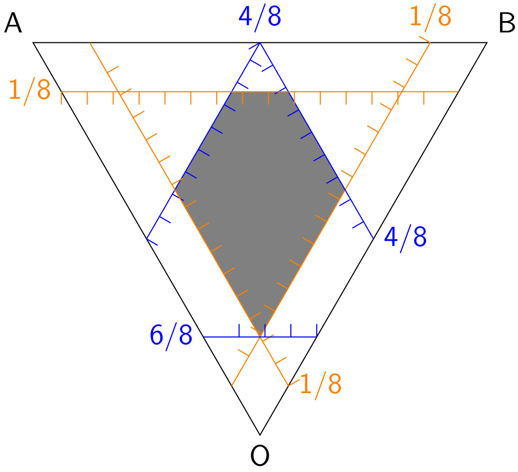
Credal sets (slot 3–4)

Acceptability & Desirability (slot 4–5)

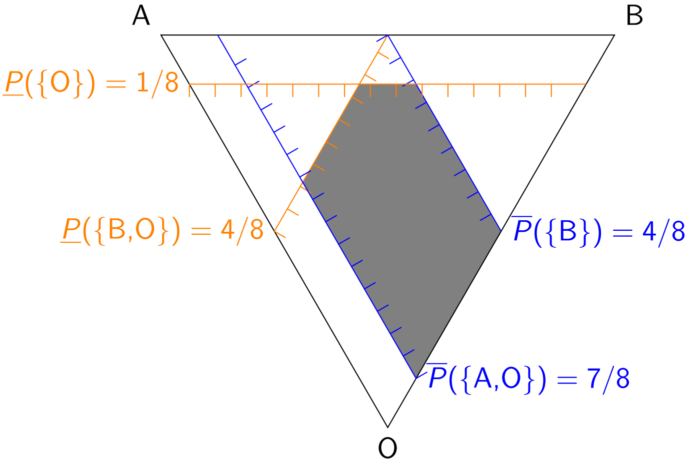
Interval expectation & probability (slot 5–6)

- Representation
- Reasoning
- Representation (continued)
- Learning
- Multivariate lower expectations

Probability interval example



Interval probability example



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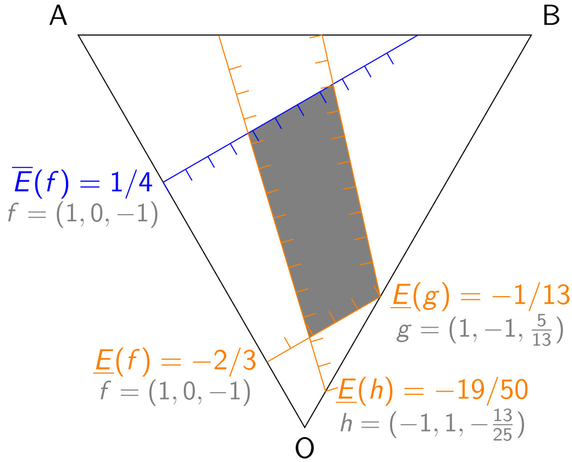

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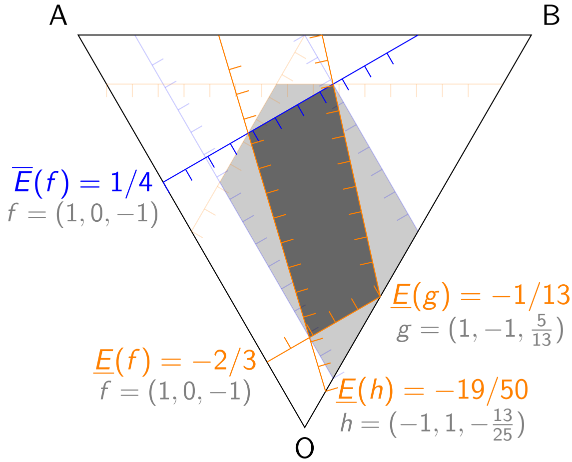

Basic setup, axioms, and terminology

Basic setup of **the theory of coherent interval probability & expectation**:

- Random variable X
- Set of outcomes $\mathcal{X}$
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Axioms

The **lower and upper expectation** operators $\underline{E}$ and $\overline{E}$, must be:

?

Eliciting lower & upper expectations with a relaxed betting game

- Two players:
  - *subject*: gambler, states acceptable exchanges, their '*assessment*'
  - *bookie*: chooses combination of acceptable exchanges
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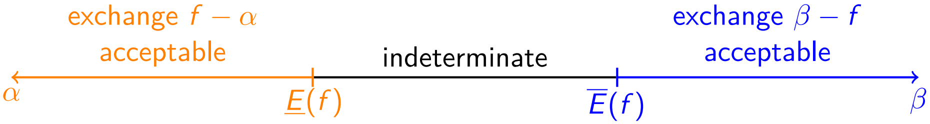

Assessment as a set of acceptable gambles

- The subject states $\underline{E}(f)$ for f and $\overline{E}(g)$ for g
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$$\mathcal{A} = \left\{ f - \alpha \cdot 1_{\mathcal{X}} : \alpha < \underline{E}(f) \right\} \cup \left\{ \beta \cdot 1_{\mathcal{X}} - g : \beta > \overline{E}(g) \right\}$$

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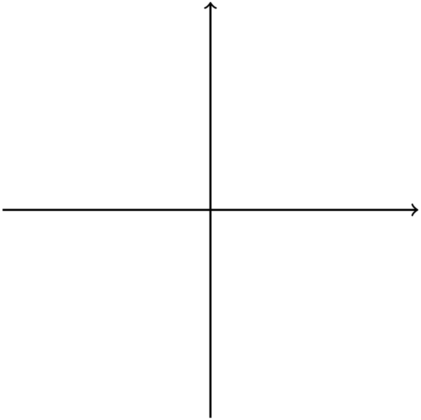

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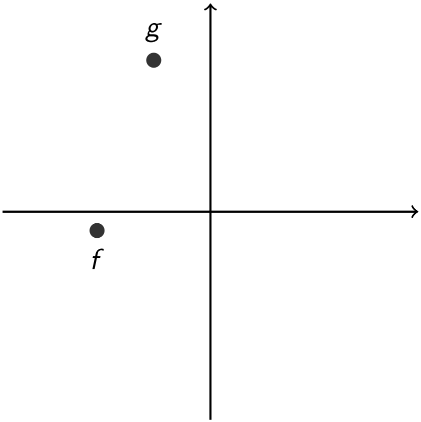

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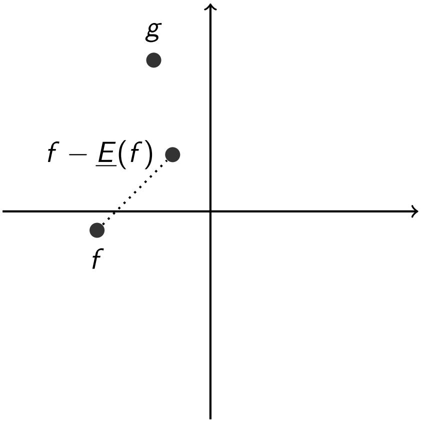

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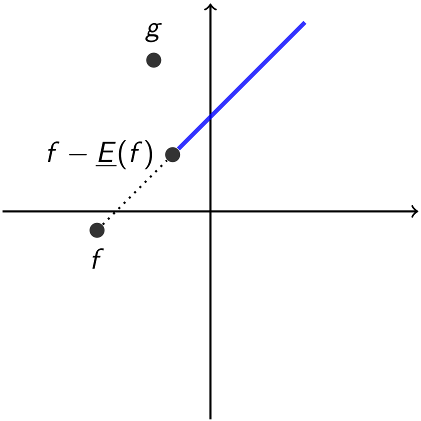

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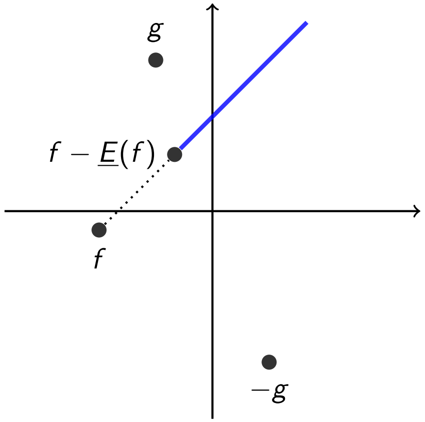

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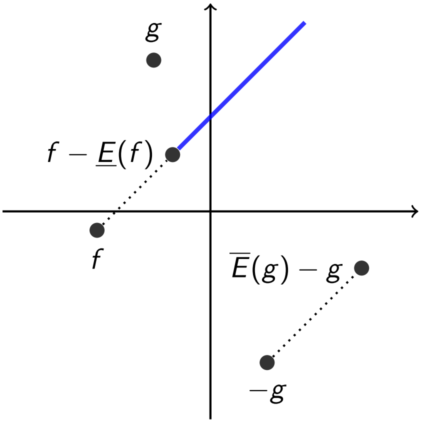

Assessment as a set of acceptable gambles

- The subject states $\underline{E}(f)$ for f and $\overline{E}(g)$ for g
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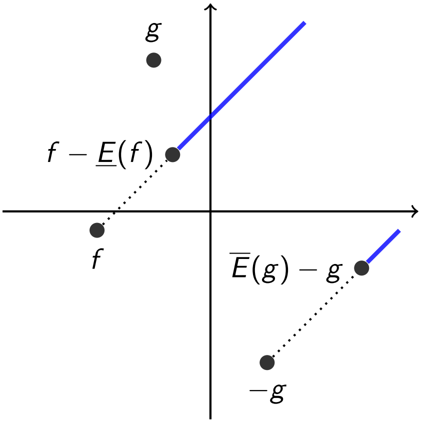

Assessment as a set of acceptable gambles & marginal gambles

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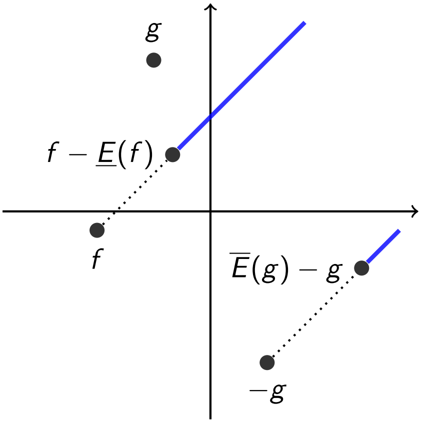
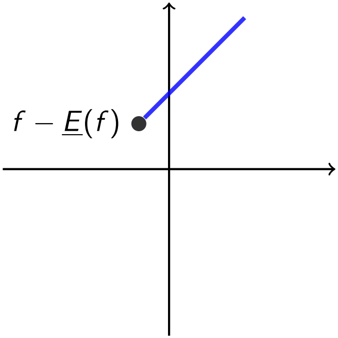


- $f - \underline{E}(f)$ and $\overline{E}(g) - g$ are called *marginal gambles*; collect them in a set $\mathcal{M}$, then

$$\mathcal{A} = \bigcup_{h \in \mathcal{M}} \{ h + \varepsilon \cdot 1_{\mathcal{X}} : \varepsilon \in \mathbb{R}_{>} \} = \mathcal{M} + \mathbb{R}_{>} \cdot 1_{\mathcal{X}}$$

Avoiding sure loss & natural extension

- We work in the language of acceptable gambles and later translate to the language of interval expectation



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- posi $\mathcal{A}$ **avoids sure loss** iff

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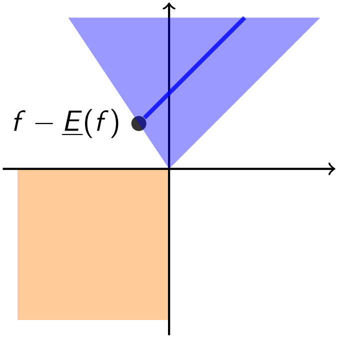

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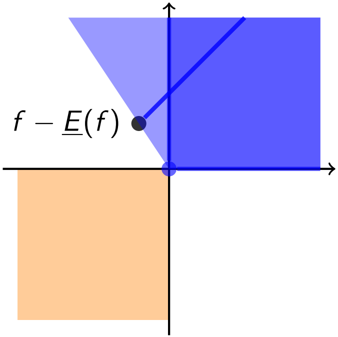
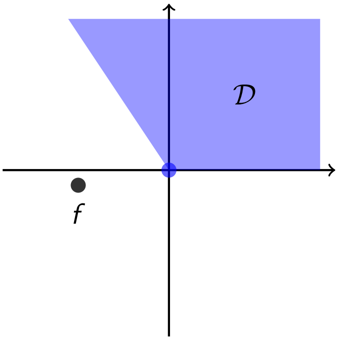
- The **natural extension** is then

$$\begin{aligned}\mathcal{E}(\mathcal{A}) &= (\operatorname{posi}\mathcal{A}+\mathcal{L}_{\geq})\cup\mathcal{L}_{\geq}\\ &= (\operatorname{posi}\mathcal{M}+\mathbb{R}_{>}\cdot 1_{\mathcal{X}}+\mathcal{L}_{\geq})\cup\mathcal{L}_{\geq}\\ &= (\operatorname{posi}\mathcal{M}+\mathcal{L}_{\gtrdot})\cup\mathcal{L}_{\geq}\end{aligned}$$



From acceptable gambles to lower & upper expectations

- Consider a coherent set of acceptable gambles $\mathcal{D}$ and a gamble f
- We infer lower and upper expectations for f using the betting game interpretation



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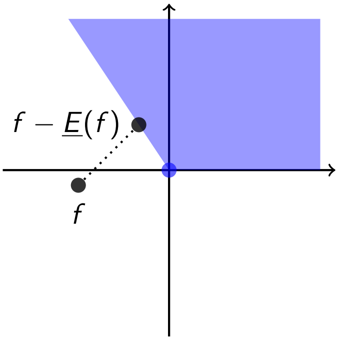

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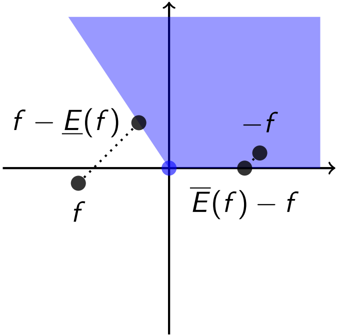

Natural extension to lower and upper expectations

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- **Upper expectation**

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\overline{E}_*(f) = \inf_{h \in \operatorname{posi}\mathcal{M} \cup \{0\}} \max(f + h) \qquad \text{(linear optimization problem over a cone)}
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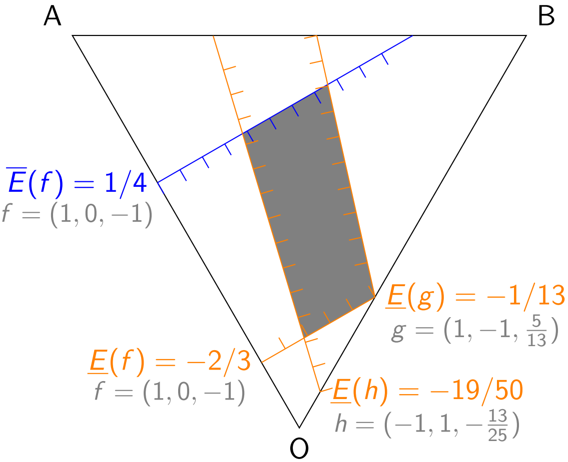
Natural extension illustration



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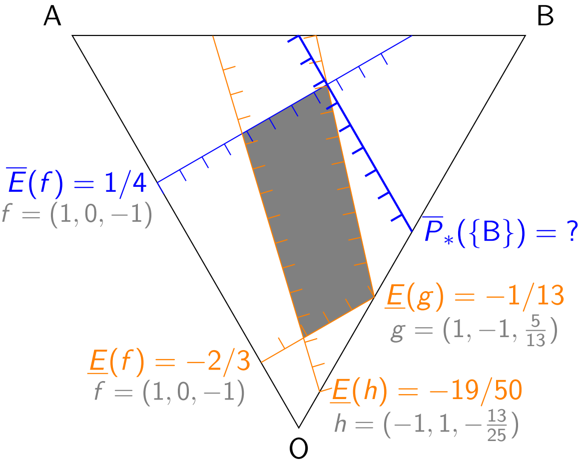

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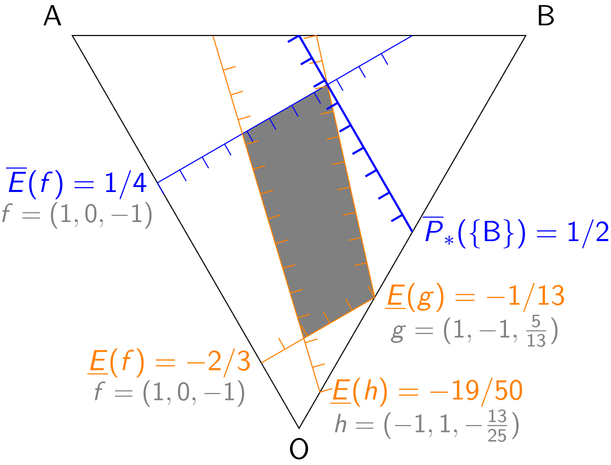

Basic setup and axioms

Basic setup of **the theory of coherent interval expectations**:

- Random variable X
- Set of outcomes $\mathcal{X}$
- Set of functions $\mathcal{F} \subseteq (\mathcal{X} \to \mathbb{R})$
- Each $f \in \mathcal{F}$ is assigned a **lower or upper expectation value**, leading to a set of marginal gambles $\mathcal{M}$

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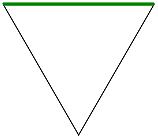
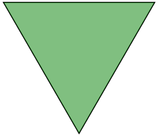
$$\inf_{h \in \operatorname{posi} \mathcal{M} \cup \{0\}} \max(h - (f - \underline{E}(f))) \geq 0$$

Special types of lower expectations

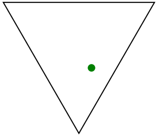
- The vacuous lower expectation $\underline{E}^S$ relative to some event $S \subseteq \mathcal{X}$ is defined by $\underline{E}^S(f) := \min_{x \in S} f(x)$ and expresses that $X \in S$ and nothing more



- The vacuous lower expectation $\underline{E}^{\mathcal{X}}$ expresses complete ignorance

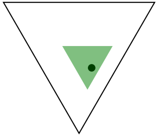


- The *linear* expectation E_p corresponding to some pmf p is also a coherent lower expectation



- The linear-vacuous lower expectation $\underline{E}_{p,\varepsilon}$ is as well:

$$\underline{E}_{p,\varepsilon}(f) := (1-\varepsilon)E_p(f) + \varepsilon \min_{x \in \mathcal{X}} f(x)$$



Some properties of coherent lower and upper expectations

Conjugacy $\overline{E}(f) = -\underline{E}(-f)$

Boundedness $\min_{x\in\mathcal{X}} f(x) \leq \underline{E}(f) \leq \overline{E}(f) \leq \max_{x\in\mathcal{X}} f(x)$

Positive homogeneity $\underline{E}(\lambda f) = \lambda \underline{E}(f)$ for all $\lambda > 0$

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Monotonicity if $f \leq g$ then $\underline{E}(f) \leq \underline{E}(g)$

Constant additivity $\underline{E}(f + \mu 1_{\mathcal{X}}) = \underline{E}(f) + \mu$ for all $\mu \in \mathbb{R}$

Mixed super/sub-linearity

$$\underline{E}(f) + \underline{E}(g) \leq \underline{E}(f+g) \leq \underline{E}(f) + \overline{E}(g) \leq \overline{E}(f+g) \leq \overline{E}(f) + \overline{E}(g)$$

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Convex mixtures if $\underline{E}^1$ and $\underline{E}^2$ are coherent, then so is $\varepsilon \underline{E}^1 + (1-\varepsilon)\underline{E}^2$ for all $0 \leq \varepsilon \leq 1$

Lower expectations on linear function spaces

If $\mathcal{F}$ is a linear space $(f, g \in \mathcal{F}$ then $af + bg \in \mathcal{F}$ for all $a, b \in \mathbb{R})$
then the axioms simplify:

Boundedness $\underline{E}(f) \geq \min_{x \in \mathcal{X}} f(x)$ for all $f \in \mathcal{F}$

Positive homogeneity $\underline{E}(\lambda f) = \lambda \underline{E}(f)$ for all $f \in \mathcal{F}$ and $\lambda > 0$

Super-linearity $\underline{E}(f + g) \geq \underline{E}(f) + \underline{E}(g)$ for all $f, g \in \mathcal{F}$

Lower envelopes

- Let Γ be some index set
- Consider a set $\{\underline{E}^\gamma : \gamma \in \Gamma\}$ of coherent lower expectations on $\mathcal{F}$
- Then its lower envelope $\underline{E}$ defined by $\underline{E}(f) := \inf_{\gamma\in\Gamma} \underline{E}^\gamma(f)$ on $\mathcal{F}$ is coherent

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Lower envelopes of credal sets

- Let $\mathcal{C}$ be a non-empty credal set
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Credal sets from lower expectations

- Let $\underline{E}$ be a lower expectation on $\mathcal{F}$ *avoiding sure loss*
- Then the credal set $\mathcal{C}^{\underline{E}} := \{p \in \mathcal{P}_X : E_p(f) \geq \underline{E}(f) \text{ for all } f \in \mathcal{F}\}$ is non-empty

The categorical prediction task

What is the color of the next ball to be drawn from an urn?

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Setup

- Outcome space $\mathcal{X}$ (of ball colors: Red, Green, Black, White,...)
- We have $n \in \mathbb{N}$ observations $\boldsymbol{x} = (x_1, \dots, x_n)$ (sequence of colors of previous balls drawn)
- Draw inferences for or make decisions related to the next observation X_{n+1} (color of next ball drawn)

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  (color of next ball drawn)

Exchangeability assumption

- *The order of the observations is irrelevant*
  (inferences should remain the same when $(\mathrm{R}, \mathrm{G}, \mathrm{R})$ or $(\mathrm{R}, \mathrm{R}, \mathrm{G})$ has been observed)
- Observed occurrence vector $\boldsymbol{n} \in \mathbb{N}^{\mathcal{X}}$ with $n_z := \sum_{k=1}^{n} \delta_{z x_k}$ and $n_S := \sum_{x \in S} n_x$
  (if $\boldsymbol{x} = (\mathrm{R}, \mathrm{G}, \mathrm{R}, \mathrm{W})$ then $n_\mathrm{R} = 2$, $n_\mathrm{G} = 1$, $n_\mathrm{W} = 1$, and $n_{\neg\mathrm{W}} = 3$)

Learning lower expectations for categorical prediction

Idea

Add epistemic uncertainty using $s > 0$ pseudo-observations (of unknown color) and predict according to the ‘observed’ frequency

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Predictive inference

- Outcome prediction: $\underline{P}_s(\{x\}|\boldsymbol{n}) = \frac{n_x}{s+n}$ and $\overline{P}_s(\{x\}|\boldsymbol{n}) = \frac{s+n_x}{s+n}$

Learning lower expectations for categorical prediction

Idea

Add epistemic uncertainty using $s > 0$ pseudo-observations (of unknown color) and predict according to the ‘observed’ frequency

Predictive inference

- Event prediction: $\underline{P}_s(S|\boldsymbol{n}) = \frac{n_S}{s+n}$ and $\overline{P}_s(S|\boldsymbol{n}) = \frac{s+n_S}{s+n}$

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- Lower expectation for $f \in \mathcal{L}$:

$$\underline{E}_s(f|\boldsymbol{n}) = \frac{n}{s+n} E_p(f) + \frac{s}{s+n} \min_{x\in\mathcal{X}} f(x) \quad \text{with } p_x := \frac{n_x}{n}$$

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Properties of $\underline{E}_s(\cdot|\boldsymbol{n})$

- linear-vacuous model; vacuous for $n = 0$; more precise with more observations
- does not depend on a specific categorization $\mathcal{X}$
- immediate prediction model of the *imprecise Dirichlet-multinomial model* ID(M)M

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Example setup

- $\boldsymbol{x} = (\mathrm{R}, \mathrm{G}, \mathrm{R}, \mathrm{W}, \mathrm{W})$
- $s = 2$

Inferences

- $\underline{P}_2(\{B, W\}|\boldsymbol{n}) = ?$
- $\overline{P}_2(\{B, W\}|\boldsymbol{n}) = ?$

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Inferences

- $\underline{P}_2(\{B, W\}|\boldsymbol{n}) = 2/7$
- $\overline{P}_2(\{B, W\}|\boldsymbol{n}) = 4/7$

Multivariate interval expectation: basic setup & basic idea

- Index set $N = \{1, \ldots, n\}$
- Multivariate variable $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$
- Set of possible outcomes $\boldsymbol{x} \in \mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$
- A **joint lower expectation** $\underline{E}^{\boldsymbol{X}}$ is specified on some set of functions $\mathcal{F} \subseteq \mathcal{L}$

Formulate an appropriate function on $\mathcal{L}$
whose lower expectation provides the desired inference;
calculate the lower expectation using natural extension

Marginal lower expectations

- A marginal lower expectation is defined for a subset of the random variables
- Let $K \subseteq N$, then
  - $\boldsymbol{X}_K := (X_k : k \in K)$ and $\boldsymbol{x}_K \in \mathcal{X}_K := \times_{k \in K} \mathcal{X}_k$
  - Notation: $\underline{E}^{\boldsymbol{X}_K}$ is the $\boldsymbol{X}_K$-marginal of the joint lower expectation $\underline{E}^{\boldsymbol{X}}$

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  - Notation: $\underline{E}^{\boldsymbol{X}_K}$ is the $\boldsymbol{X}_K$-marginal of the joint lower expectation $\underline{E}^{\boldsymbol{X}}$
- The **marginal lower expectation** for $f \in \mathcal{L}_K$ is obtained by calculating the joint natural extension of its *cylindrical extension*:

$$\underline{E}^{\boldsymbol{X}_K}(f) := \underline{E}_*^{\boldsymbol{X}}(\uparrow_{\boldsymbol{x}_{N \setminus K}} f) \quad \text{where} \quad (\uparrow_{\boldsymbol{x}_{N \setminus K}} f)(\boldsymbol{x}) := f(\boldsymbol{x}_K)$$

Marginal lower expectation example

Example setup

- Two random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
- Joint lower expectation $\underline{E}^{\boldsymbol{X}} := \underline{E}_{\boldsymbol{p},\varepsilon}$ with ε unspecified and p given on the right (in black), together with its marginals (in green)

p	0	1	p^{X_2}
$-$	$3/9$	$1/9$	$4/9$
$+$	$2/9$	$3/9$	$5/9$
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Inference

Marginal lower expectation $\underline{E}^{X_1}$

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Inference

Marginal lower expectation $\underline{E}^{X_1}$

Solution

$$\begin{aligned}\underline{E}_{p,\varepsilon}(\uparrow_{\mathcal{X}_2} f) &= (1-\varepsilon)E_p(\uparrow_{\mathcal{X}_2} f) + \varepsilon \min_{\boldsymbol{x}\in\mathcal{X}}(\uparrow_{\mathcal{X}_2} f)(x_1, x_2) \\ &= (1-\varepsilon)E_{p^{X_1}}(f) + \varepsilon \min_{x_1\in\mathcal{X}_1} f(x_1) \\ &= \underline{E}_{p^{X_1},\varepsilon}(f)\end{aligned}$$

Conditional lower expectations

- A conditional lower expectation is determined by
  - assuming some event is known to be true, or specifically
  - assuming some random variables take some given values: $\boldsymbol{X}_K = \boldsymbol{x}_K$, with $K \subset N$
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- The **conditional lower expectation** for $f \in \mathcal{L}_{N\setminus K}$ is the maximum acceptable buying price for the corresponding *called-off gamble*:

$$\underline{E}^{\boldsymbol{X}_{N\setminus K}}(f|\boldsymbol{X}_K = \boldsymbol{x}_K) := \max\left\{\alpha \in \mathbb{R} : \underline{E}^{\boldsymbol{X}}_*\left(\uparrow_{\boldsymbol{x}_K}\left(f - \alpha \cdot 1_{\mathcal{X}_{N\setminus K}}\right) \cdot 1_{\boldsymbol{X}_K = \boldsymbol{x}_K}\right) \geq 0\right\}$$

  if $\underline{P}(\boldsymbol{X}_K = \boldsymbol{x}_K) > 0$, otherwise $\underline{E}^{\boldsymbol{X}_{N\setminus K}}(\cdot|\boldsymbol{X}_K = \boldsymbol{x}_K) := \underline{E}^{\mathcal{X}_{N\setminus K}}$

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if $\underline{P}(\boldsymbol{X}_K = \boldsymbol{x}_K) > 0$, otherwise $\underline{E}^{\boldsymbol{X}_{N\setminus K}}(\cdot|\boldsymbol{X}_K = \boldsymbol{x}_K) := \underline{E}^{\mathcal{X}_{N\setminus K}}$

This is called *conditioning by natural extension* or the *generalized Bayes's rule*

Conditional lower expectation example

Example setup

- Two random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
- Joint lower expectation $\underline{E}^{\boldsymbol{X}} := \underline{E}_{p,\varepsilon}$ with ε unspecified and p given on the right (in black), together with its marginals (in green)

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Inference

Conditional lower expectation $\underline{E}^{X_1}(\cdot|X_2 = +)$

Solution (start)

$$\underline{E}^{X_1}(f|X_2 = +) = \max\Big\{\alpha \in \mathbb{R} : \underline{E}_{p,\varepsilon}\big(\uparrow_{X_2} (f - \alpha \cdot 1_{\mathcal{X}_1}) \cdot 1_{X_2=+}\big) \geq 0\Big\}$$

Conditional lower expectations example

Solution (continuation)

$$\underline{E}^{X_1}(f|X_2 = +) = \max\left\{\alpha \in \mathbb{R} : \underline{E}_{p,\varepsilon}\left(\uparrow_{X_2} (f - \alpha \cdot 1_{\mathcal{X}_1}) \cdot 1_{X_2=+}\right) \geq 0\right\}$$

Conditional lower expectations example

Solution (continuation)

$$\underline{E}^{X_1}(f|X_2=+)=\max\Big\{\alpha\in\mathbb{R}:\underline{E}_{\boldsymbol{p},\varepsilon}\big(\uparrow_{X_2}(f-\alpha\cdot 1_{\mathcal{X}_1})\cdot 1_{X_2=+}\big)\geq 0\Big\}$$

$$=\max\Big\{\alpha\in\mathbb{R}:(1-\varepsilon)E_p\big(\uparrow_{X_2}(f-\alpha\cdot 1_{\mathcal{X}_1})\cdot 1_{X_2=+}\big)+\varepsilon\min_{\boldsymbol{x}\in\mathcal{X}}(f(x_1)-\alpha)\cdot 1_{X_2=+}(x_2)\geq 0\Big\}$$

Conditional lower expectations example

Solution (continuation)

$$\underline{E}^{X_1}(f|X_2=+) = \max\left\{\alpha \in \mathbb{R} : \underline{E}_{\boldsymbol{p},\varepsilon}\left(\uparrow_{X_2}\left(f-\alpha\cdot 1_{\mathcal{X}_1}\right)\cdot 1_{X_2=+}\right) \geq 0\right\}$$

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Conditional lower expectations example

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$$= \max\left\{\alpha\geq\min_{x_1\in\mathcal{X}_1}f(x_1) : (1-\varepsilon)\tfrac{5}{9}\sum_{x_1\in\mathcal{X}_1}p_{x_1}^{X_1|+}\left(f(x_1)-\alpha\right)+\varepsilon\left(\min_{x_1\in\mathcal{X}_1}f(x_1)-\alpha\right)\geq 0\right\}$$

Conditional lower expectations example

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$$\underline{E}^{X_1}(f|X_2=+) = \max\Big\{\alpha \in \mathbb{R} : \underline{E}_{p,\varepsilon}\big(\uparrow_{X_2}(f-\alpha\cdot 1_{\mathcal{X}_1})\cdot 1_{X_2=+}\big) \geq 0\Big\}$$

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$$\text{with } \delta = \frac{\varepsilon}{(1-\varepsilon)\frac{5}{9}+\varepsilon}$$

Conditional lower expectations example

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Epistemic irrelevance & Epistemic independence

Consider $\boldsymbol{X} = (X_1, X_2)$ and a joint lower expectation $\underline{E}^{\boldsymbol{X}}$ on $(\mathcal{X} \to \mathbb{R})$

Epistemic irrelevance

- X_2 is *epistemically irrelevant* to X_1 iff $\underline{E}^{X_1}(\cdot|X_2 = x_2) := \underline{E}^{X_1}$ for all $x_2 \in \mathcal{X}_2$
- Epistemic irrelevance of X_2 to X_1 does not imply epistemic irrelevance of X_1 to X_2

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Epistemic independence

X_1 and X_2 are called *epistemically independent* iff

- X_2 is epistemically irrelevant to X_1
- X_1 is epistemically irrelevant to X_2

Epistemically independent natural extension

Setup

- Consider two random variables X_1 and X_2
- Marginal lower expectations $\underline{E}^{X_1}$ on $\mathcal{F}_1 \subseteq \mathcal{L}_1$ and $\underline{E}^{X_2}$ on $\mathcal{F}_2 \subseteq \mathcal{L}_2$

Natural extension of epistemically independent marginals

- Creating a *joint* lower expectation from epistemically independent marginals is not done as typically with a product

Epistemically independent natural extension

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Natural extension of epistemically independent marginals

- Creating a *joint* lower expectation from epistemically independent marginals is not done as typically with a product
- Natural extension of the implied assessment is used:

$$\underline{E}^{\boldsymbol{X}}(f) = (\underline{E}^{X_1} \boxtimes \underline{E}^{X_2})(f) := \sup_{g_1, g_2 \in \mathcal{L}} \min_{\boldsymbol{x} \in \mathcal{X}} \Big(f(\boldsymbol{x}) - \big(g_1(\boldsymbol{x}) - \underline{E}_*^{X_1}(g_1(X_1, x_2))\big) - \big(g_2(\boldsymbol{x}) - \underline{E}_*^{X_2}(g_2(x_1, X_2))\big) \Big)$$

for all $f \in \mathcal{L}$

Introduction to imprecise probabilities

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